Oscillating Black-Scholes formula

An analytical solution

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Introduction

- Parabolic partial differential equations with piecewise constant coefficients (PPDEPCC) have several useful applications
- PPDEPCC can be used to solve the volatility calibration problem
- PPDEPCC naturally arise in such diverse areas as mathematical modeling of brain tumor growth
- PPDEPCC describe percolation of medicine administered through skin, etc.
- Probabilistically, PPDEPCC describe oscillating Brownian motion and other interesting processes

Oscillating Brownian motion

- We wish to compute $H(t, X, k)$, which solves the heat equation with piecewise constant thermal conductivity and a single wave initial condition

$$H_t - \frac{1}{2}\sigma^2(X) H_{XX} = 0, \qquad H(0, X, k) = e^{ikX},$$

where

$$\sigma(X) = \sigma_0 \mathbb{I}_{X<\xi} + \sigma_1 \mathbb{I}_{X\geq\xi},$$

and $\mathbb{I}_{.}$ is the indicator function.

- We write the solution on the same way:

$$H(t, X) = H_0(t, X)\,\mathbb{I}_{X<\xi} + H_1(t, X)\,\mathbb{I}_{X\geq\xi}.$$

Without loss of generality, we assume that $\xi = 0$.

Oscillating Brownian motion

- It can be shown that

$$H_1(t,X,k) = \mathfrak{F}\left(\frac{X}{Y_1}, ikY_1\right) + \frac{2\sigma_1}{(\sigma_1+\sigma_0)}\mathfrak{F}\left(-\frac{X}{Y_1}, -ikY_0\right) + \frac{(\sigma_0-\sigma_1)}{(\sigma_1+\sigma_0)}\mathfrak{F}\left(-\frac{X}{Y_1}, ikY_1\right).$$

- Similarly,

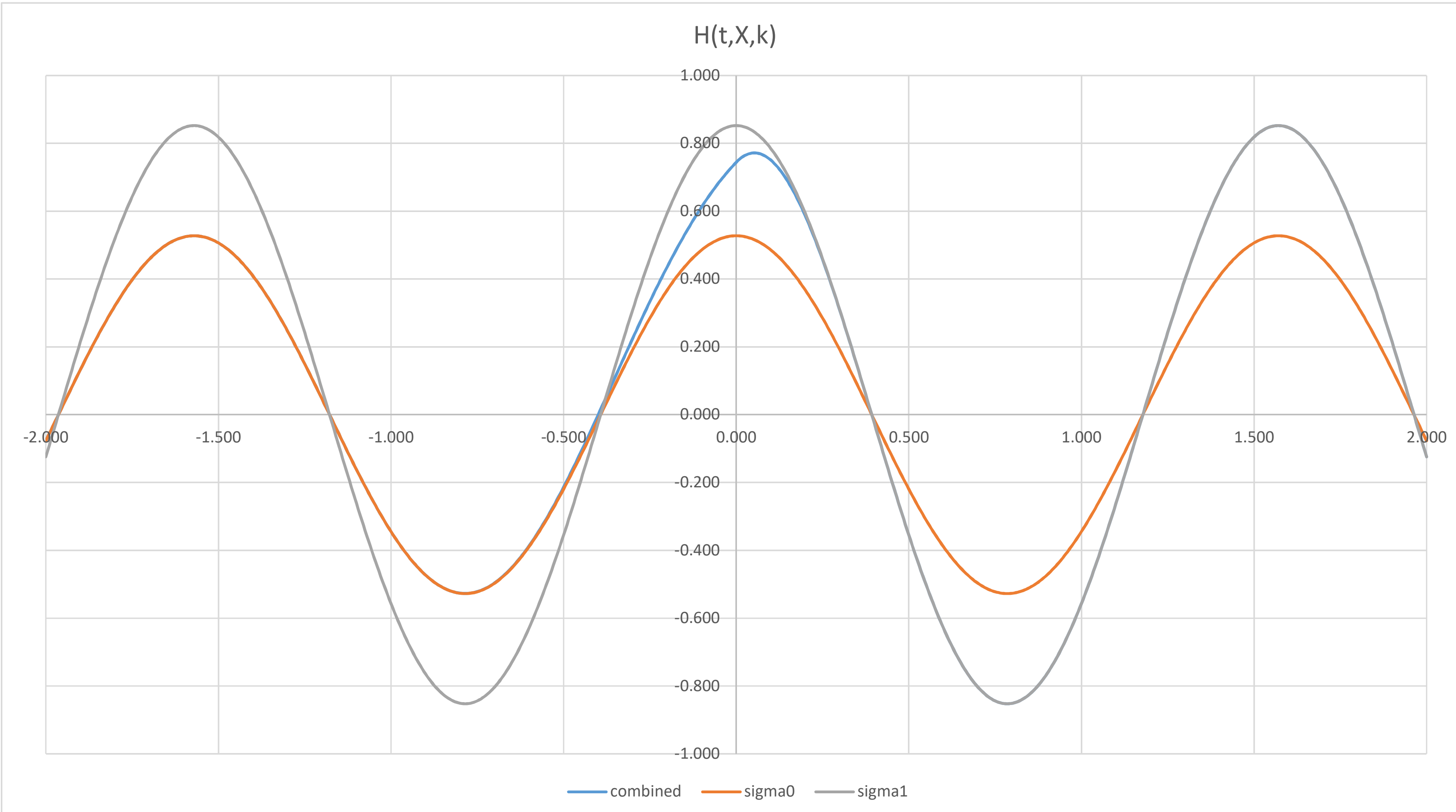
$$H_0(t,X,k) = \mathfrak{F}\left(-\frac{X}{Y_0}, -ikY_0\right) + \frac{2\sigma_0}{(\sigma_1+\sigma_0)}\mathfrak{F}\left(\frac{X}{Y_0}, ikY_1\right) + \frac{(\sigma_1-\sigma_0)}{(\sigma_1+\sigma_0)}\mathfrak{F}\left(\frac{X}{Y_0}, -ikY_0\right).$$

- Here $Y_j = \sqrt{\sigma_j t}$, and $\mathfrak{F}$, $\mathfrak{M}$ are given by

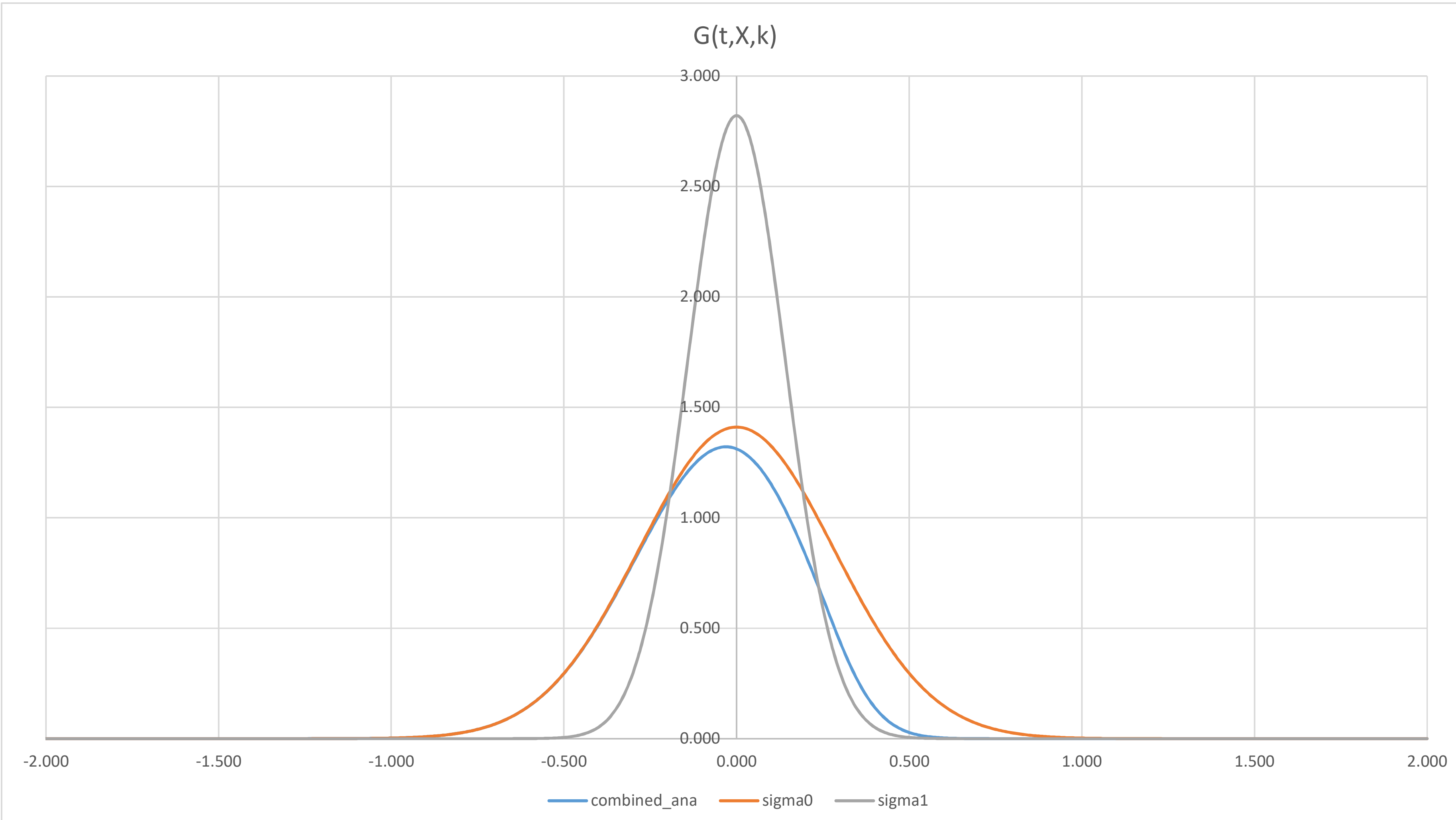
$$\mathfrak{F}(a,b) = e^{ab}\mathfrak{M}(a+b), \qquad \mathfrak{M}(x+iy) = e^{-\frac{y^2}{2}}\mathfrak{N}(x+iy).$$

- The combined formula has the form

$$H_j(t, X, k) = \mathfrak{F}\left(\left(-1\right)^{1-j}\frac{X}{Y_j}, \left(-1\right)^{1-j} ikY_j\right)$$
$$+\frac{2\sigma_j}{(\sigma_1+\sigma_0)}\mathfrak{F}\left(\left(-1\right)^{j}\frac{X}{Y_j}, \left(-1\right)^{j} ikY_{1-j}\right)$$
$$+\frac{(\sigma_{1-j}-\sigma_j)}{(\sigma_1+\sigma_0)}\mathfrak{F}\left(\left(-1\right)^{j}\frac{X}{Y_1}, \left(-1\right)^{1-j} ikY_1\right).$$



H(t,X,k)
1.000
0.800
0.600
0.400
0.200
0.000
-0.200
-0.400
-0.600
-0.800
-1.000
-2.000
-1.500
-1.000
-0.500
0.000
0.500
1.000
1.500
2.000
combined
sigma0
sigma1



G(t,X,k)
3.000
2.500
2.000
1.500
1.000
0.500
0.000
-2.000
-1.500
-1.000
-0.500
0.000
0.500
1.000
1.500
2.000
combined_ana
sigma0
sigma1

Oscillating Black-Scholes formula

- We start with Dupire's equation (1994) for the price of a call option $C(t,K)$:

$$C_t - \frac{1}{2}\sigma^2(K)K^2 C_{KK} = 0, \qquad C(0,K) = (S-K)_+,$$

$$\sigma(K) = \sigma_0 \mathbb{I}_{K<\xi S} + \sigma_1 \mathbb{I}_{K \geq \xi S}.$$

- We wish to rewrite this equation as Lipton's equation (2002). To this end, we define the covered call

$$\tilde{C}(t,K) = S - C(t,K), \qquad \tilde{C}(0,K) = \min(K,S).$$

normalize independent and dependent variables as follows:

$$X = \ln\left(\frac{K}{S}\right), \quad \tilde{C} = e^{\frac{1}{2}X} S D(t,X),$$

and write the pricing problem for D in the form

$$D_t - \frac{1}{2}\sigma^2\left(D_{XX} - \frac{1}{4}D\right) = 0, \qquad D(0,X) = e^{-\frac{1}{2}|X|}.$$

Oscillating Black-Scholes formula

- We introduce the Carson-Laplace transform Ψ of D,

$$\Psi(\lambda, X) = \lambda \int_0^\infty e^{-\lambda t} D(t, X)\, dt,$$

- An inhomogeneous ODE for Ψ has the form

$$\lambda \Psi - \frac{1}{2}\sigma^2 \left(\Psi_{XX} - \frac{1}{4}\Psi \right) = \lambda e^{-\frac{1}{2}|X|}.$$

- The volatility is specified as follows

$$\sigma(X) = \sigma_0 \mathbb{I}_{X<\eta} + \sigma_1 \mathbb{I}_{X \geq \eta},$$

where $\eta = \ln(\xi)$. To be concrete, we assume that $\xi > 1$, so that $\eta > 0$.

Oscillating Black-Scholes formula

- We can represent the inverse Carson-Laplace transform

$$\mathcal{A}\left(t,X\right)=\mathcal{L}^{-1}\left(\Phi\left(\lambda,X\right)/\lambda\right),$$

as follows

$$\begin{gathered}\mathcal{A}\left(t,X,\eta,\sigma_0,\sigma_1\right)=\mathcal{A}_{1-s}\left(t,X,\eta,\sigma_0,\sigma_1\right)\mathbb{I}_{X<\eta}\\ +\mathcal{A}_s\left(t,X,\eta,\sigma_0,\sigma_1\right)\mathbb{I}_{X\geq\eta}.\end{gathered}$$

Oscillating Black-Scholes formula

- Here $s = (sign(\eta) + 1)/2$, and

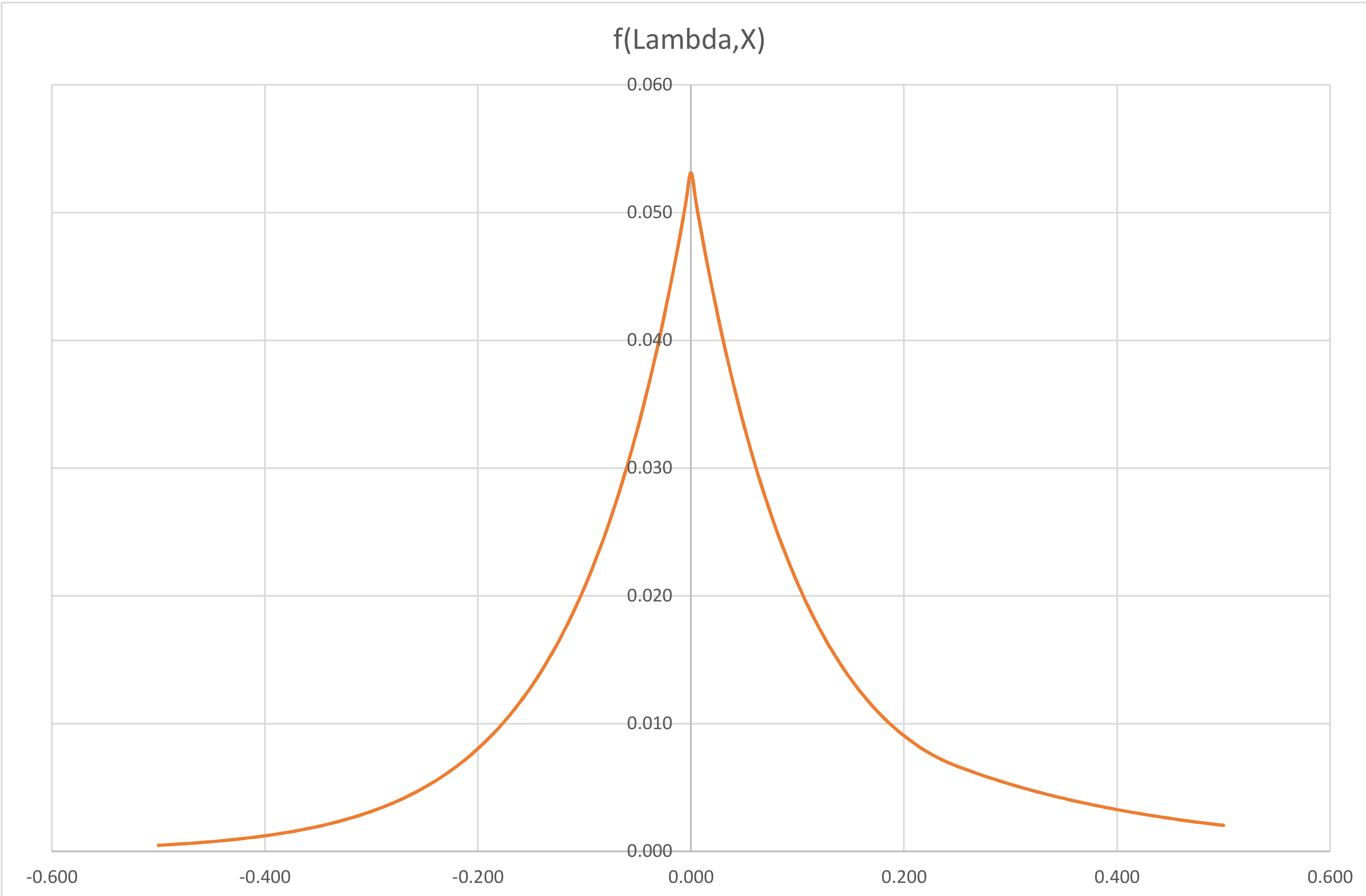
$$
\begin{aligned}
&\mathcal{A}_0(t, X, \eta, \sigma_0, \sigma_1) \\
&= \Theta^{(-1,1)}\left(\sqrt{\sigma_{1-s}^2 t}, |X|\right) \\
&-\Theta^{(-1,1)}\left(\sqrt{\sigma_{1-s}^2 t}, (2s-1)(2\eta - X)\right) \\
&+\frac{\sigma_s^2}{\left(\sigma_s^2 - \sigma_{1-s}^2\right)}\bar{\Theta}^{(1,2)}\left(\sqrt{\sigma_{1-s}^2 t}, (2s-1)(2\eta - X)\right) \\
&-\frac{\sigma_0^2\sigma_1^2}{4\left(\sigma_s^2 - \sigma_{1-s}^2\right)}\Theta^{(1,1)}\left(\sqrt{\sigma_s^2 t}, 0\right) * \Theta^{(0,1)}\left(\sqrt{\sigma_{1-s}^2 t}, (2s-1)(2\eta - X)\right), \\
&\mathcal{A}_1(t, X, \eta, \sigma_0, \sigma_1) \\
&= \frac{\sigma_0^2\sigma_1^2}{4\left(\sigma_s^2 - \sigma_{1-s}^2\right)}\left(\Theta^{(1,1)}\left(\sqrt{\sigma_{1-s}^2 t}, |\eta|\right) * \Theta^{(0,1)}\left(\sqrt{\sigma_s^2 t}, (2s-1)(X - \eta)\right)\right. \\
&\left.-\Theta^{(0,1)}\left(\sqrt{\sigma_{1-s}^2 t}, |\eta|\right) * \Theta^{(1,1)}\left(\sqrt{\sigma_s^2 t}, (2s-1)(X - \eta)\right)\right).
\end{aligned}
$$

Oscillating Black-Scholes formula

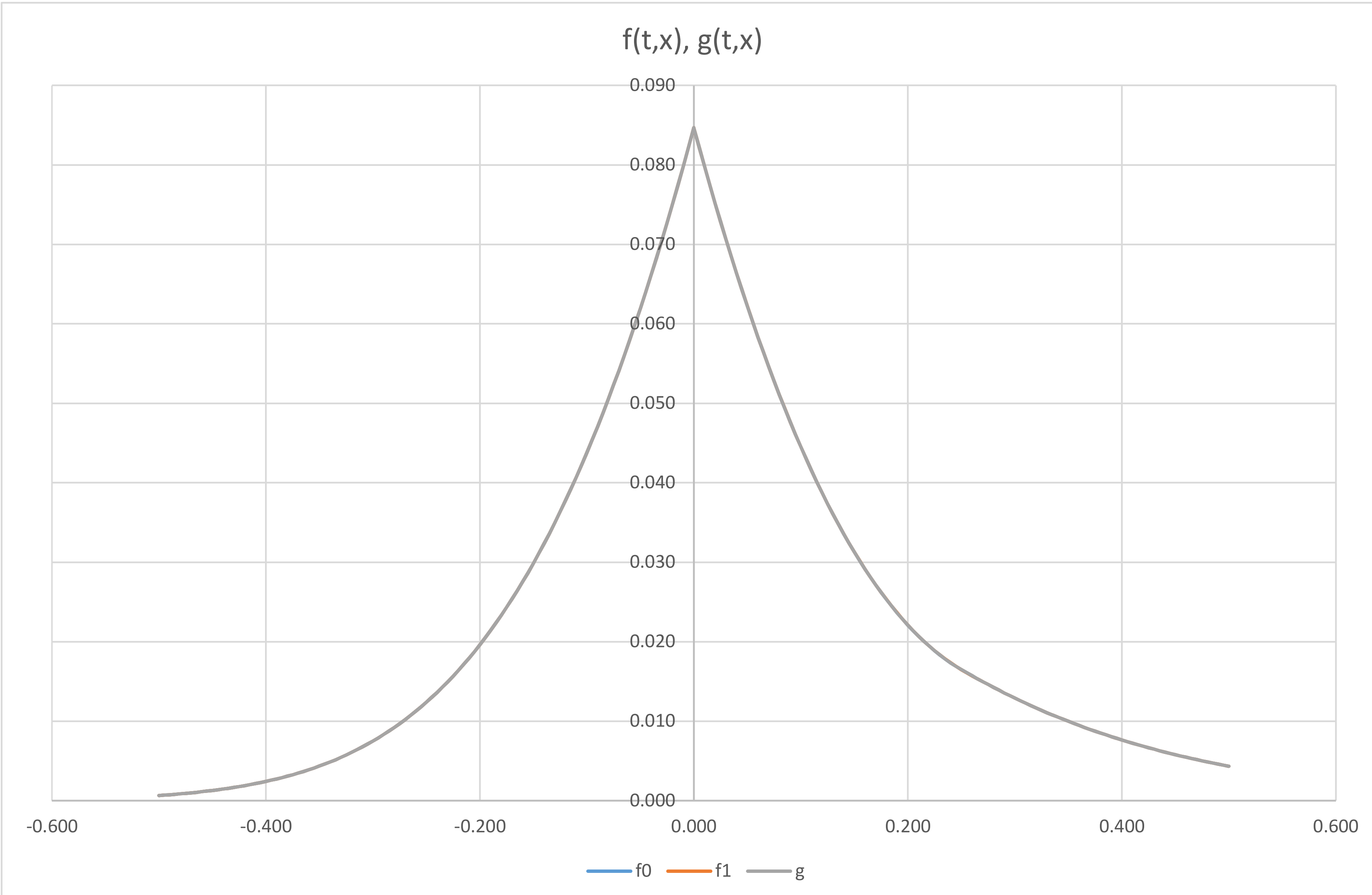
- The corresponding Θ have the form

$$\begin{gathered}
\bar{\Theta}^{(1,2)}(Y,Z) = \Theta^{(-1,1)}(Y,Z) + \frac{1}{4}Y^2\Theta^{(1,1)}(Y,Z) - \frac{1}{2}Z\Theta^{(0,1)}(Y,Z) \\
\Theta^{(1,1)}(Y,Z) = \mathfrak{F}\left(-\frac{Z}{Y},\frac{Y}{2}\right) - \mathfrak{F}\left(-\frac{Z}{Y},-\frac{Y}{2}\right) + \frac{4}{Y}e^{-\frac{Y^2}{8}}\mathfrak{n}\left(\frac{Z}{Y}\right), \\
\Theta^{(0,1)}(Y,Z) = \mathfrak{F}\left(-\frac{Z}{Y},\frac{Y}{2}\right) + \mathfrak{F}\left(-\frac{Z}{Y},-\frac{Y}{2}\right), \\
\Theta^{(-1,1)}(Y,Z) = \mathfrak{F}\left(-\frac{Z}{Y},\frac{Y}{2}\right) - \mathfrak{F}\left(-\frac{Z}{Y},-\frac{Y}{2}\right),
\end{gathered}$$

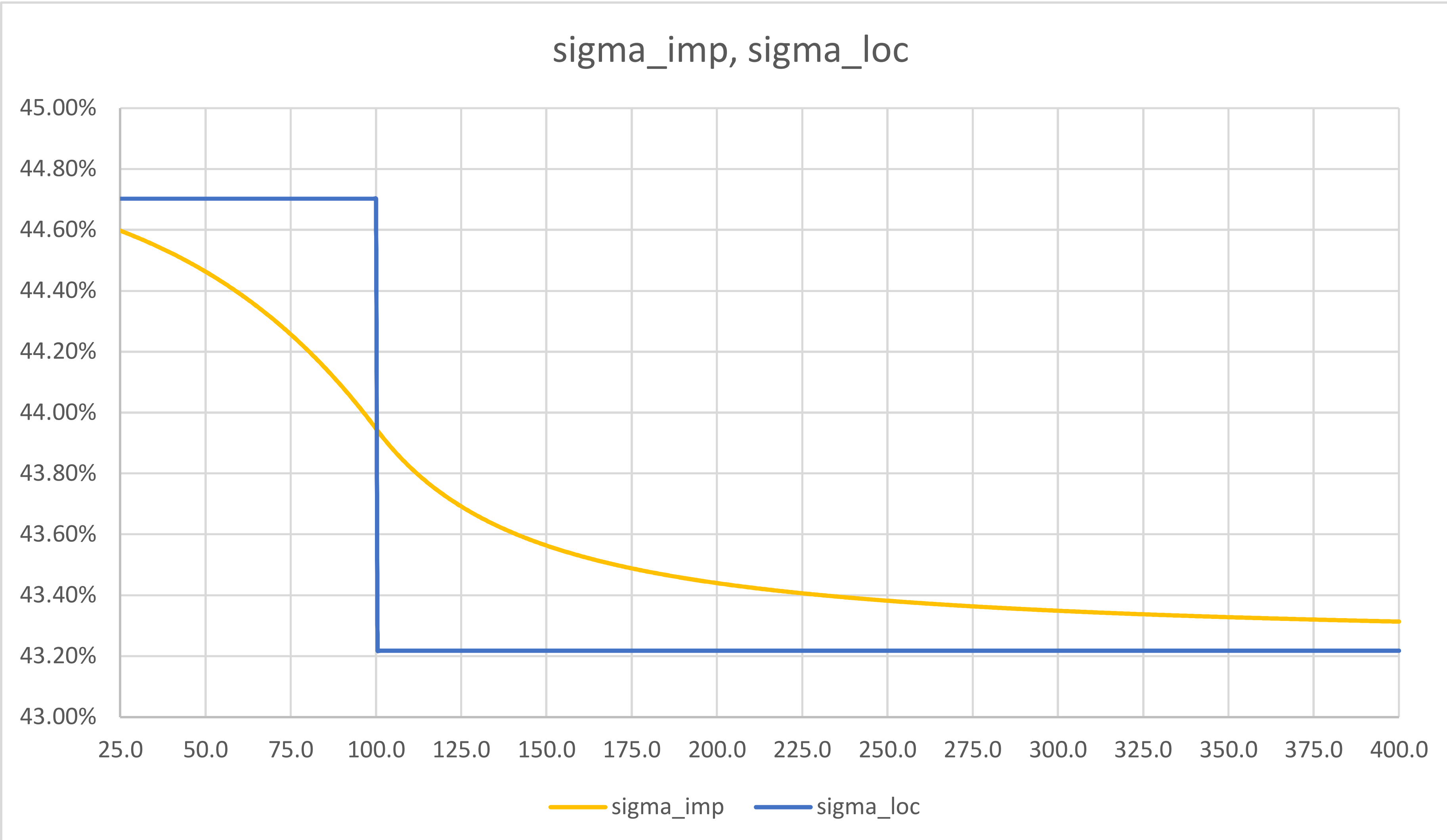
- Here $Y = \sqrt{\sigma^2 t}$.

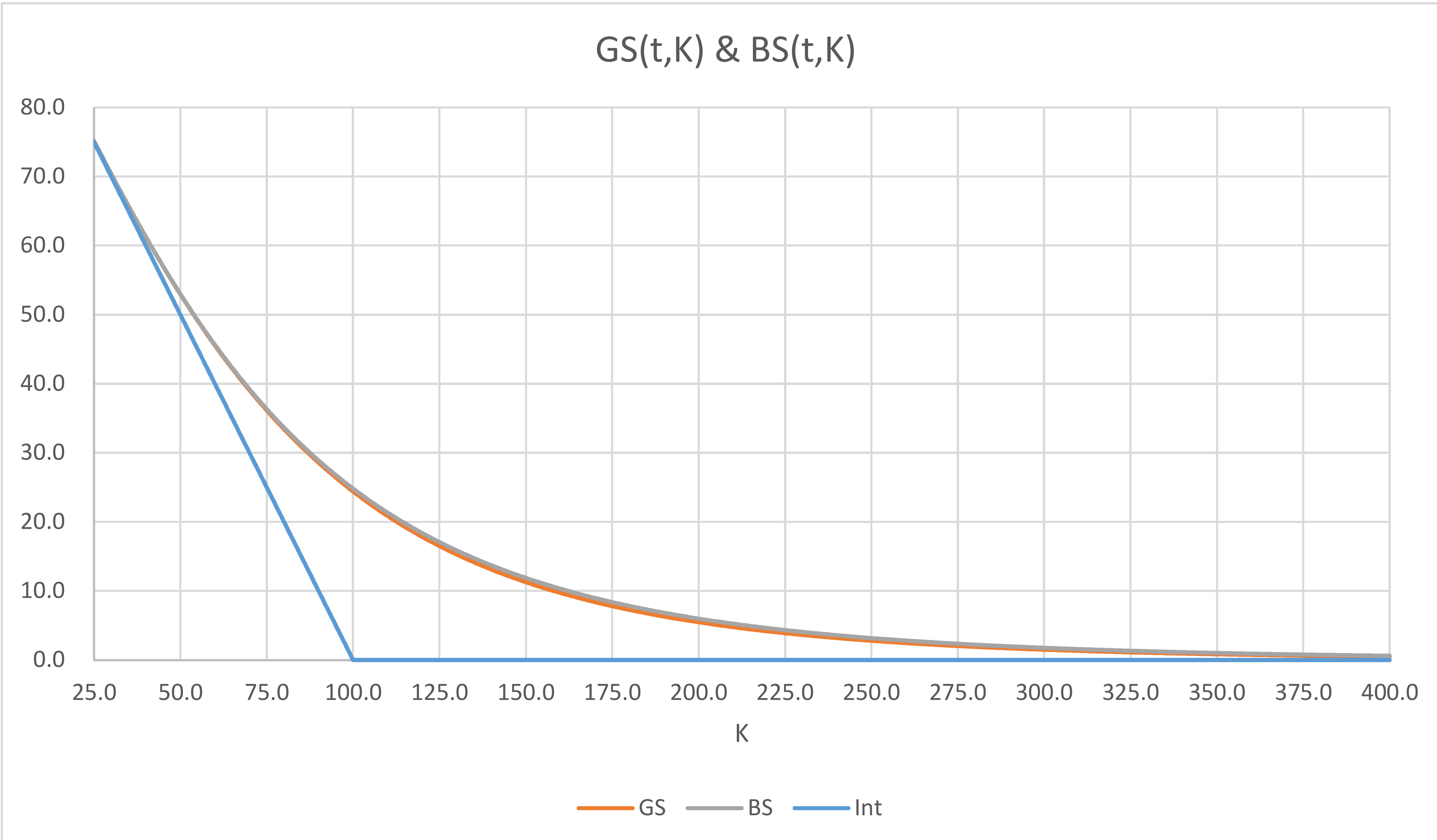


f(Lambda,X)
0.060
0.050
0.040
0.030
0.020
0.010
0.000
-0.600
-0.400
-0.200
0.000
0.200
0.400
0.600



f(t,x), g(t,x)
0.090
0.080
0.070
0.060
0.050
0.040
0.030
0.020
0.010
0.000
-0.600
-0.400
-0.200
0.000
0.200
0.400
0.600
f0
f1
g


sigma_imp, sigma_loc
45.00%
44.80%
44.60%
44.40%
44.20%
44.00%
43.80%
43.60%
43.40%
43.20%
43.00%
25.0
50.0
75.0
100.0
125.0
150.0
175.0
200.0
225.0
250.0
275.0
300.0
325.0
350.0
375.0
400.0
sigma_imp
sigma_loc



GS(t,K) & BS(t,K)
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
0.0
25.0
50.0
75.0
100.0
125.0
150.0
175.0
200.0
225.0
250.0
275.0
300.0
325.0
350.0
375.0
400.0
K
GS
BS
Int

The multi-layer case

- We can also solve Lipton's equation (2002) with multi-variate volatility,

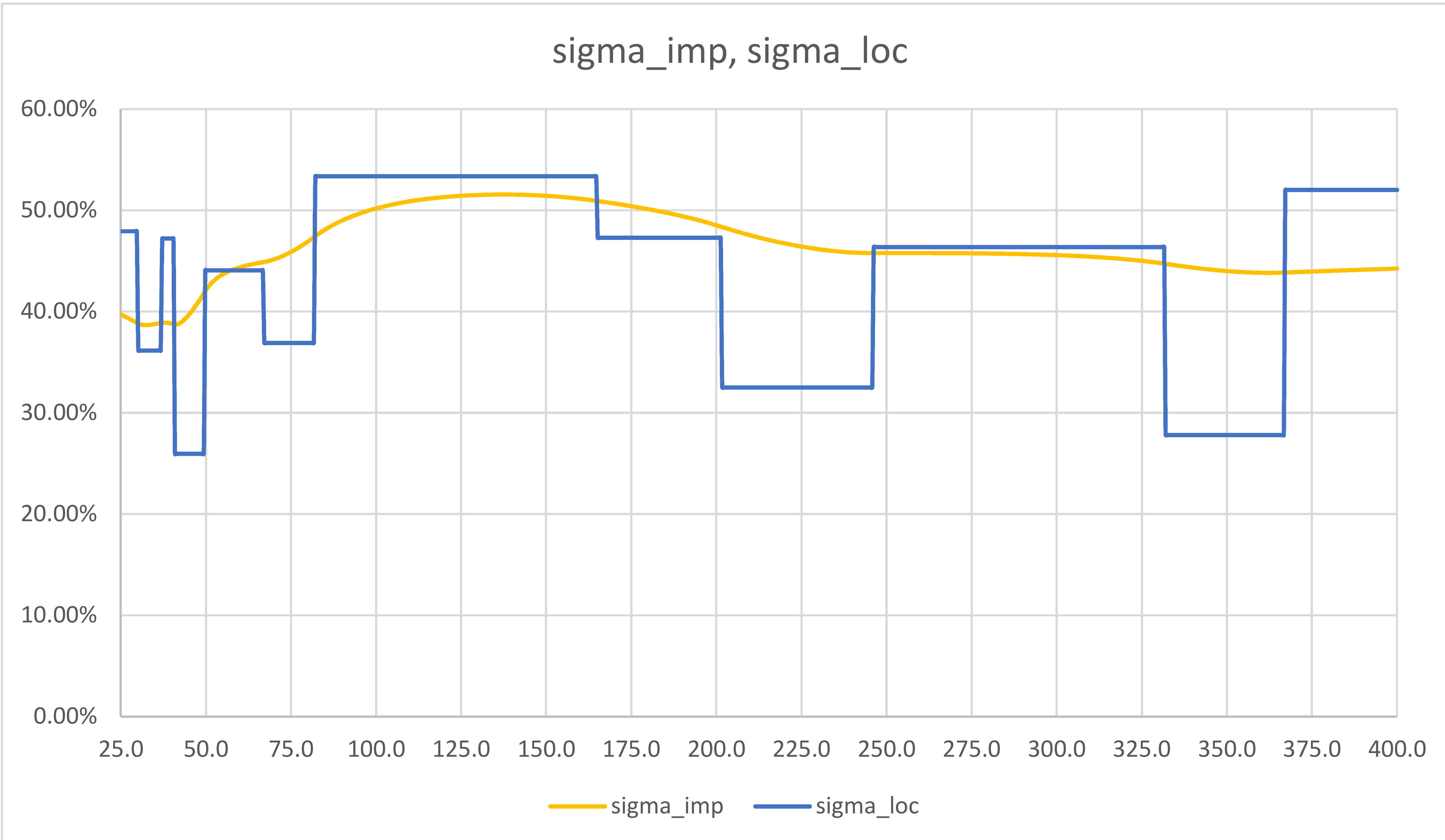
$$D_t - \frac{1}{2}\sigma^2 \left(D_{XX} - \frac{1}{4} D \right) = 0, \qquad D(0, X) = e^{-\frac{1}{2}|X|}.$$

- Here

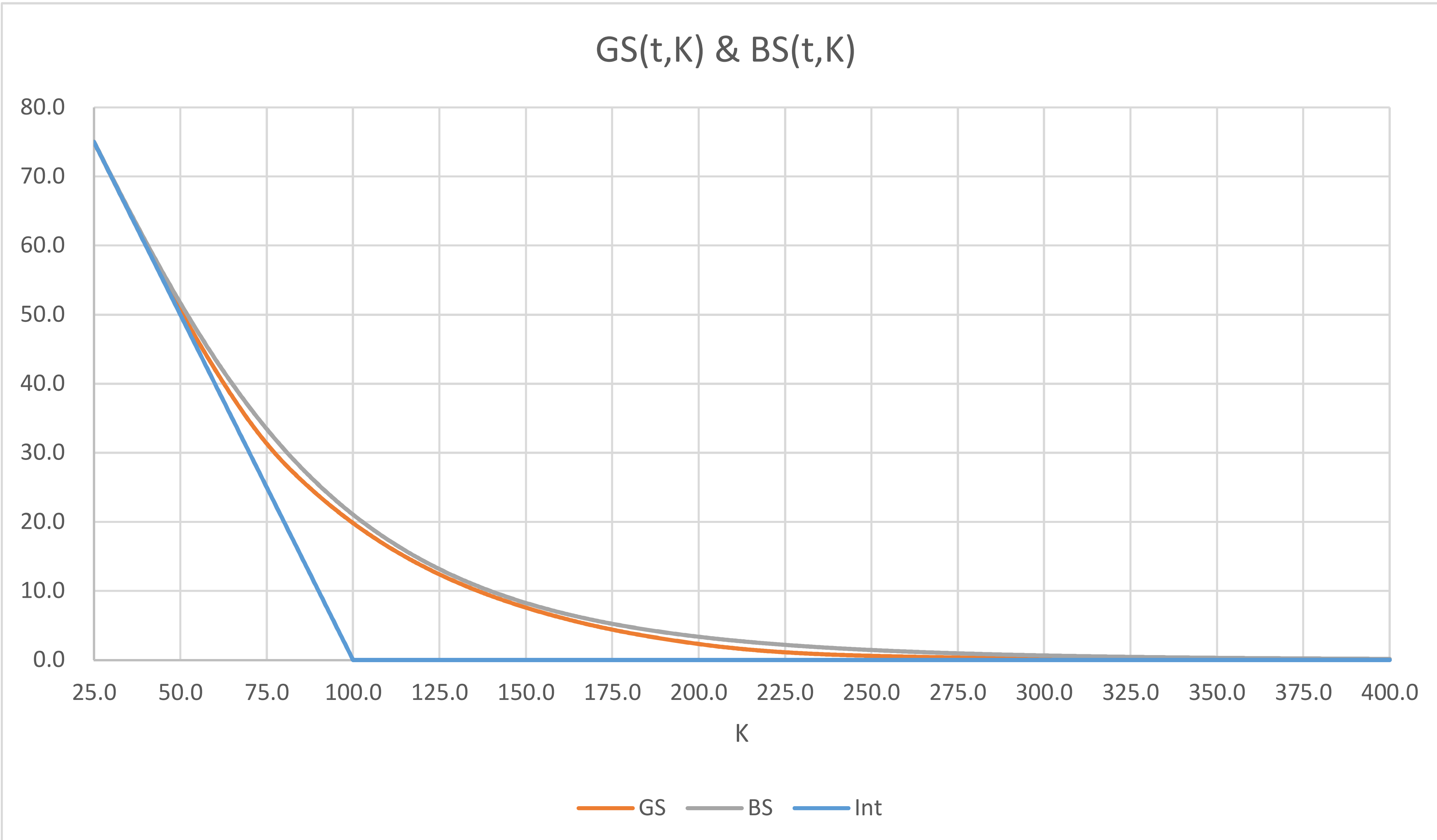
$$\sigma(X) = \begin{cases} \bar{\sigma}_n, & X \in \bar{I}_n = \left(\bar{\eta}_{n+1}, \bar{\eta}_n\right], \\ \sigma_n, & X \in I_n = \left(\eta_n, \eta_{n+1}\right]. \end{cases}$$

- The points where the local volatility jumps have the form,

$$\begin{gathered} \bar{\eta}_{\bar{N}+1} = -\infty < \bar{\eta}_{\bar{N}} < \ldots < \bar{\eta}_n < \ldots < \bar{\eta}_1 < \bar{\eta}_0 = 0 \\ 0 = \eta_0 < \eta_1 < \ldots < \eta_n < \ldots < \eta_N < \eta_{N+1} = \infty. \end{gathered}$$



sigma_imp, sigma_loc
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
25.0
50.0
75.0
100.0
125.0
150.0
175.0
200.0
225.0
250.0
275.0
300.0
325.0
350.0
375.0
400.0
sigma_imp
sigma_loc


GS(t,K) & BS(t,K)
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
0.0
25.0
50.0
75.0
100.0
125.0
150.0
175.0
200.0
225.0
250.0
275.0
300.0
325.0
350.0
375.0
400.0
K
GS
BS
Int

Bibliography

- J. Keilson, J. Wellner, 1978
- B. Dupire, 1994
- A. Lipton, 2002
- J. Andreasen, B. Huge, 2011
- A. Lipton, A. Sepp, 2011
- A. Lipton, 2016
- A. Itkin, A. Lipton, 2017
- A. Lejay, P. Pigato, 2017

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