Rev Deriv Res (2017) 20:15–46
DOI 10.1007/s11147-016-9123-1

Structural default model with mutual obligations

Andrey Itkin[1,2] · **Alexander Lipton**[1,3,4]

Published online: 6 June 2016

Abstract In this paper we consider mutual obligations in an interconnected bank system and analyze their impact on the joint and marginal survival probabilities for individual banks. We also calculate prices of the corresponding credit default swaps and first-to-default swaps. To make the role of mutual obligations more transparent, we develop a simple structural default model with banks' assets driven by correlated multidimensional Brownian motion with drift. We calculate closed form expressions for many quantities of interest and use them for the efficient model calibration. We demonstrate that mutual obligations have noticeable impact on the system behavior.

Keywords 2D structural default model · Mutual obligations · Joint and marginal survival probabilities · CDS and First-to-Default swap prices

---

The views expressed in this paper represent those of the authors and not necessarily those of Bank of America.

✉ Andrey Itkin
aitkin@nyu.edu

Alexander Lipton
alexlipt@mit.edu

1 Bank of America, New York, NY, USA

2 New York University, School of Engineering, Brooklyn, NY, USA

3 Massachusetts Institute of Technology, Connection Science, Cambridge, MA, USA

4 Courant Institute, New York University, New York, NY, USA

1 Introduction

The structural default framework is widely used for assessing credit risk of the corporate debt. In its simplest form, it was introduced in the seminal work (Merton 1974), and was further extended in various papers, see a survey in Lipton and Sepp (2011) and references therein. In contrast to reduced-form models (see, e.g., Bielecki and Rutkowski 2004) structural default models suffer from the curse of dimensionality when the number of counterparties grows; however, these models provide a more detailed and financially meaningful description of the default event for a typical firm.

Inspired by Webber and Willison (2011), in Itkin and Lipton (2014) we extended the structural default framework by taking into account the fact that banks have mutual liabilities to each other. Taking this effect into consideration is very important in order to accurately analyze credit worthiness of individual banks and the banking network as a whole. For instance, large mutual liabilities imply that adverse shock to a bank is rapidly transmitted to the entire system, with severe implications for its stability (David and Lehar 2014). The authors of David and Lehar (2014) indicate that renegotiations between highly interconnected banks facilitate mutual private sector bailouts to lower the need for government bailouts. The relative size of mutual liabilities compared to total liabilities is quite significant. For instance, the relative fraction of interbank loans is 12 % in the EU, 8.5 % for Canada (David and Lehar 2014), and 4.5 % for US (as per Economic Research website of the Federal Reserve Bank of St. Louis).

An extended Merton model with mutual liabilities and continuous default monitoring can be built by combining correlated Merton balance sheet models calibrated by using observed bank equity returns, and analyzing potential clearing of interbank liabilities in the spirit of Eisenberg and Noe (2001). In Itkin and Lipton (2014) we assumed that banks' assets are driven by correlated Lévy processes with idiosyncratic and common components and developed a novel method for solving the corresponding partial pseudo differential equation, which made the problem of computing joint and marginal survival probabilities computationally tractable. We discussed the impact of mutual liabilities on the system as a whole, and gave numerical examples illustrating its importance.

Obviously, the knowledge of the joint and marginal survival probabilities is important for successful calibration of the model to the prices of credit default swaps (CDSs) and first-to-default swaps (FTD). Since the general case is fairly complicated, here we restrict ourselves to the special case when banks' assets are driven by the correlated N-dimensional Wiener process with drift. Then in the 2D case we obtain explicit expressions for several quantities of interest including the joint and marginal survival probabilities as well as CDSs and FTD prices. Despite the fact that the model under consideration does not incorporate jumps, it is still interesting, especially because it can be solved analytically; in addition, it provides a natural link to the analytical framework considered in Lipton and Sepp (2011), Lipton and Savescu (2014).

The rest of the paper is organized as follows. In Sect. 2 we propose a model for the general case of N banks. In Sects. 3 and 4 we present the governing equations and describe Green's function based approach for calculating the joint and marginal survival probabilities for two banks with mutual obligations. In Sects. 5 and 6 we calculate the prices of CDS and FTD contracts, and present results of our numerical experiments. We also validate our results by comparing analytical solutions with numerical solutions obtained by using a finite difference algorithm described in Itkin and Lipton (2014). In Sect. 7 we discuss the calibration of the model and present some additional numerical results. Sect. 8 concludes.

2 The model

Consider a set of N banks with external assets and liabilities, which are denoted by $A_i, L_i, i = 1, \ldots, N$, and interbank assets and liabilities which are denoted by L_{ji}, $j = 1, \ldots, N$. In other words, L_{ij} is the amount which the i-th bank owes to the j-th bank, etc. Thus, total assets, liabilities and capital of the i-th bank have the form

$$\widetilde{A}_i = A_i + \sum_{j \neq i} L_{ji}, \quad \widetilde{L}_i = L_i + \sum_{j \neq i} L_{ij}, \quad C_i = \widetilde{A}_i - \widetilde{L}_i = A_i - \lambda_i^{=},$$

where

$$\lambda_i^{=} = L_i + \sum_{j \neq i} \left(L_{ji} - L_{ij} \right). \tag{1}$$

For simplicity, we assume that the corresponding dynamics are governed by the SDEs of the form

$$\begin{aligned} dA_{i,t} &= rA_{i,t}dt + \sigma_i A_{i,t} dW_{i,t}, \\ dL_{i,t} &= rL_{i,t}dt, \quad dL_{ij,t} = rL_{ij,t}dt, \end{aligned} \tag{2}$$

subject to the initial conditions $A_{i,0} = A_i(0)$, $L_{i,0} = L_i(0)$, $L_{ij,0} = L_{ij}(0)$, so that $L_{i,t}, L_{ij,t}$ are deterministic functions of the time t.[1] In Eq. (2) r is the risk-free rate, σ_i is the volatility of the i-th asset (which is assumed to be constant), and $W_{i,t}$ are components of the N-dimensional Wiener process with the correlation matrix ρ_{ij}. These assumptions can be generalized in a variety of ways, which will be discussed elsewhere.

We assume that all the liabilities (both external and interbank) are settled at a certain maturity $T > 0$. Thus, the i-th bank defaults at $t = T$ if $\widetilde{A}_{i,T} < \widetilde{L}_i(T)$, or, equivalently, if $A_{i,T} < \lambda_i^{=}(T)$. Below we denote the default time of the ith bank by τ_i.

The k-th bank defaults at $\tau_k < T$ We describe defaults at intermediate times $0 < \tau_i < T$ in the spirit of Black and Cox (1976) by assuming that the i-th bank defaults at time τ_i provided that

$$A_i(\tau_i) \le \lambda_i^{<}(\tau_i) \equiv R_i \left[L_i(\tau_i) + \sum_{j \neq i} L_{ij}(\tau_i) \right] - \sum_{j \neq i} L_{ji}(\tau_i), \tag{3}$$

[1] Accordingly, further we will denote them as $L_i(t)$, $L_{ij}(t)$.

where $0 \le R_i \le 1$ is the recovery rate for $0 < t < T$, which is assumed to be constant. We emphasize that in this setup the default boundary is discontinuous at $t = T$, because R_i experiences a jump at this point from its value R_i at $t < T$ to 1 at $t = T$ (and so $\lambda_i^{<}$ transforms to $\lambda_i^{=}$).

These default boundaries are valid provided that no other bank defaults before $t = T$. If we assume that the k-th bank is the first to default at time $\tau_k < T$, then we are left with a set of $N - 1$ surviving banks. At the default time τ_k the assets and liabilities of the i-th bank, $i \neq k$, assume the form

$$\widetilde{A}_i(\tau_k) = A_i(\tau_k) + \sum_{j \neq i, j \neq k} L_{ji}(\tau_k) + R_k L_{ki}(\tau_k),$$
$$\widetilde{L}_i(\tau_k) = L_i(\tau_k) + \sum_{j \neq i, j \neq k} L_{ij}(\tau_k) + L_{ik}(\tau_k).$$

We assume that for surviving banks mutual liabilities stay the same, while their external liabilities jump according to the rule

$$L_i(\tau_k) \to \bar{L}_i(\tau_k) \equiv L_i(\tau_k) + L_{ii}(\tau_k) - R_k L_{ki}(\tau_k).$$

Surviving banks' capital naturally takes a hit

$$C_i(\tau_k) \to \bar{C}_i(\tau_k) \equiv C_i(\tau_k) - (1 - R_k) L_{ki}(\tau_k).$$

Thus, each default reduces the set of surviving banks and modifies the corresponding default boundaries as follows:

$$\tilde{\lambda}_{ik}(t) = \begin{cases} \lambda_{ik}^{<}(t), & t < T, \\ \lambda_{ik}^{=}(T), & t = T, \end{cases} \quad i \neq k$$
$$\lambda_{ik}^{<}(t) = R_i \left[L_i(t) + L_{ik}(t) - R_k L_{ki}(t)\right] + \sum_{j \neq i, j \neq k} \left[R_i L_{ij}(t) - L_{ji}(t)\right],$$
$$\lambda_{ik}^{=}(T) = L_i(t) + L_{ik}(t) - R_k L_{ki}(t) + \sum_{j \neq i, j \neq k} \left[L_{ij}(t) - L_{ji}(t)\right]. \tag{4}$$

As an example, consider $N = 2$. Assume that the k-th bank defaults first, $k \in [1, 2]$. Then for the remaining bank $\bar{k} \equiv 3 - k$, the default boundary is given by Itkin and Lipton (2014)[2]

$$\tilde{\lambda}_{\bar{k}}(t) = \begin{cases} R_{\bar{k}} \left(L_{\bar{k}} + L_{\bar{k}k} - R_k L_{k\bar{k}}\right), & \tau_k \le t < T, \\ L_{\bar{k}} + L_{\bar{k}k} - R_k L_{k\bar{k}}, & \tau_k < t = T. \end{cases} \tag{5}$$

[2] In this case we omit the second index of λ_{ik} as the defaulted bank is determined uniquely.

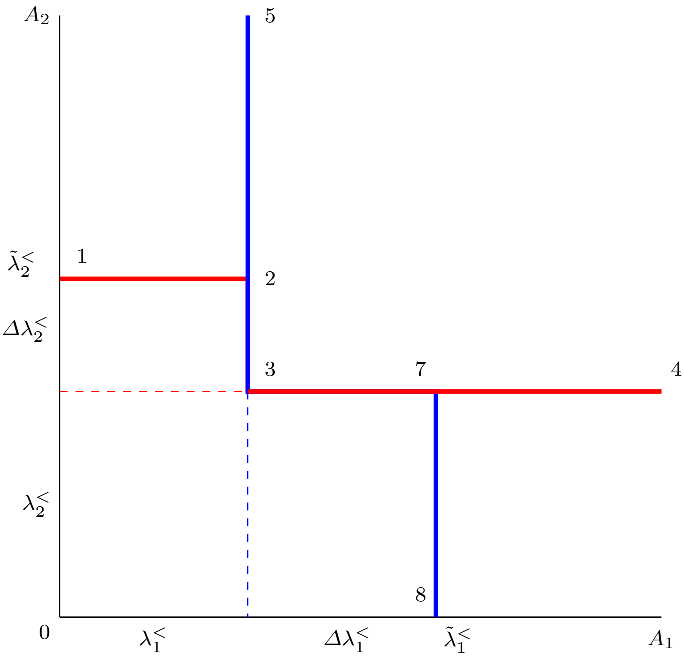

Fig. 1 Default boundaries of two banks with and without mutual liabilities at $t < T$

It is clear that

$$\Delta\lambda_{\bar{k}} = \tilde{\lambda}_{\bar{k}} - \lambda_{\bar{k}} = \begin{cases} \left(1 - R_{\bar{k}} R_k\right) L_{k\bar{k}} \equiv \Delta\lambda_{\bar{k}}^{<}, & \tau_k \le t < T, \\ (1 - R_k)\, L_{k\bar{k}} \equiv \Delta\lambda_{\bar{k}}^{=}, & t = T, \end{cases}$$

and $\Delta\lambda_{\bar{k}} \ge 0$.

To make these definitions more transparent the corresponding boundaries are represented in Fig. 1. Here, if there are no defaults, we have a rectangular computational domain which lies above the piece-wise constant line 5–3–4. If the bank 2 defaults, this domain transforms to that which lies to the right of the line 5–3–7–8. If the bank 1 defaults, the domain transforms to that which lies above the line 1–2–3–4.

The i-th bank defaults at $\tau_i = T$ In this case the definition of $\tilde{\lambda}_i$ in Eq. (5) should be changed. Indeed, if assets of the i-th bank breach below its liabilities at some time before maturity, the bank has some period of time to recover, unless it breaches below the level $\tilde{\lambda}_i^<$. At this level the bank's counterparties don't believe anymore in its ability to recover, and it defaults. Obviously, at $t = T$ the bank doesn't have time to recover. Therefore, at the most it can pay to its obligors the current amount of money in hands, i.e. the total value of the bank assets[3] which is a fraction $\gamma_i,\ 0 < \gamma_i \le 1$,

[3] We consider an idealistic situation when all bank's assets upon default can be immediately converted to cash with no delay and further losses.

of its liabilities. Accordingly, in the spirit of Eisenberg and Noe (2001), to determine default boundaries we need to find the vector $\boldsymbol{\gamma} = \{\gamma_i\}, \ 0 \le \gamma_i \le 1, \ i \in [1, N]$ which solves the following piece-wise linear problem in the unit cube:

$$\min\left\{A_i(T) + \sum_{j\neq i} \gamma_j L_{ji}(T), \ L_i(T) + \sum_{i\neq j} L_{ij}(T)\right\} = \gamma_i \left(L_i(T) + \sum_{j\neq i} L_{ij}(T)\right). \tag{6}$$

Introducing new non-dimensional variables $a_i = A_i(T)/\tilde{L}_i(T), \ l_{ji} = L_{ji}(T)/\tilde{L}_i(T)$ the problem given in Eq. (6) can be re-written in the form

$$\min\left\{a_i + \sum_{j\neq i} \gamma_j l_{ji}, \ 1\right\} = \gamma_i. \tag{7}$$

It is clear that $\gamma_i = 1$ (so that the i-th bank survives) if $a_i + \sum_{j\neq i} \gamma_j l_{ji} \ge 1$. And $\gamma_i < 1$ otherwise, so that the i-th bank defaults. This description suggests that defaults in the interlinked set of banks can happen outright, when

$$A_i\,(T) < \lambda_i^{=}\,(T)\,,$$

and through contagion, when

$$L_i(T) + \sum_{j\neq i} \left[L_{ij}\,(T) - \gamma_j L_{ji}\,(T)\right] > A_i\,(T) \ge \lambda_i^{=}\,(T)\,.$$

Eq. (7) can be uniquely solved. A brief discussion is given in Appendix 1.

Accordingly, in this case we change the definition of $\lambda_i^{=}(T)$ at $\tau_i = T$ to

$$\tilde{\lambda}_{i,T} = L_i(T) + \sum_{i\neq j} L_{ij}(T) - \sum_{j\neq i} \tilde{R}_{j,T}(\boldsymbol{\gamma}) L_{ji}(T) \tag{8}$$

where

$$\tilde{R}_{j,T}(\boldsymbol{\gamma}) = \min\left[1, \ \frac{A_j(T) + \sum_{i\neq j} \gamma_i L_{ij}(T)}{L_j(T) + \sum_{i\neq j} \gamma_i L_{ji}(T)}\right], \quad \boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_N]. \tag{9}$$

It follows that the default boundary $\tilde{\lambda}_{i,T}$ piece-wise linearly depends on all $A_j(T), \ j \in [1, N], \ j \neq i$. In particular, let $N = 2$ and $\tau_2 = T$, hence when $A_2(T) = 0$ we have from Eq. (8)

$$\tilde{\lambda}_{1,T} = \frac{\triangle}{L_2 + L_{21}}, \quad \triangle = L_1 L_2 + L_{12} L_2 + L_1 L_{21}.$$

Therefore,

$$\tilde{\lambda}_1^{=}\Big|_{\tau_2 < T} - \tilde{\lambda}_{1,T} = L_{21}(\tilde{R}_{2,T}(1) - R_2) = L_{21}\left(\frac{L_{12}}{L_2 + L_{21}} - R_2\right).$$

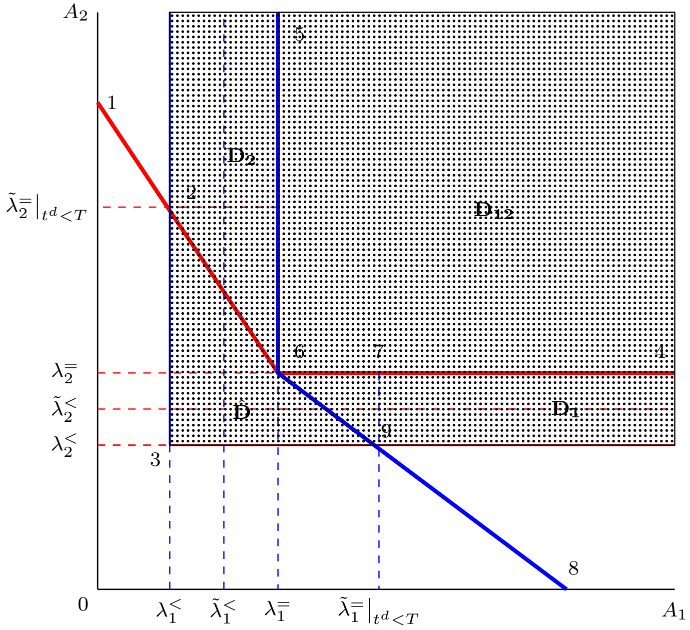

Fig. 2 Default boundaries of two banks with and without mutual liabilities at $t = T$. *The dot pattern marks* the whole computational domain $\mathfrak{D}$

This behavior is illustrated in Fig. 2. Since $\tau_j = T$, and, thus, $R_j = 1$, from Eq. (5) we have $\tilde{\lambda}_i^{=} = \lambda_i^{=}$, or $\Delta\lambda_i = 0$. Therefore, when $A_2(T)$ grows from 0 to $\lambda_2^{=}$, the default boundary $\lambda_{1,T}$ moves from $\dfrac{\triangle}{L_2 + L_{21}}$ to $\lambda_1^{=}$ along the line 8–9–6.

At point 6 the default boundary $\tilde{\lambda}_{1,T}$ transforms to $\tilde{\lambda}_1^{=} = \lambda_1^{=}$, and, further, doesn't depend on $A_2(T)$ when the latter increases. This occurs at the point $A_2^{=}(T) = \lambda_2^{=}$. Thus, the whole default boundary of the first bank in Fig. 2 can be seen as a line passing through the points 8–9–6–5. Similarly, for the second bank the default boundary in Fig. 2 can be seen as a line passing through the points 1–2–6–4.

Also in Fig. 2, $\mathbf{D}_{12}$ is the domain where both banks don't default, $\mathbf{D}_1$ where the second bank defaults while the first one does not, and $\mathbf{D}_2$ where the first bank defaults while the first one does not; $\mathfrak{D}$ is the whole computational domain marked by the dot pattern, and in the domain $\mathbf{D} = \mathfrak{D}\backslash\mathbf{D}_{12}$ both banks default.

As always, it is useful to describe the evolution of the set of banks under consideration in terms of non-dimensional variables. To this end, we introduce the average volatility $\omega \equiv \left(\prod_{i-1}^{N} \sigma_i\right)^{1/N}$, and define

$$\bar{t} = \omega^2 t, \quad \bar{T} = \omega^2 T, \quad X_{i,\bar{t}} = \frac{\omega}{\sigma_i} \ln\left(\frac{A_{i,\bar{t}}}{\lambda_i^{<}(\bar{t})}\right), \quad \xi_i = -\frac{1}{2}\frac{\sigma_i}{\omega}.$$

The corresponding dynamics of $\bar{X}_{i,\bar{t}}$ is governed by the SDE:

$$d\bar{X}_{i,\bar{t}} = \xi_i d\bar{t} + dW_{i,\bar{t}}, \tag{10}$$

while the default conditions now transform to $X_i \le \mu_i$, with μ_i defined as

$$\mu_i\left(\bar{t}\right) = \begin{cases} \mu_i^{<} \equiv 0, & \bar{t} < \bar{T}, \\ \mu_i^{=} \equiv \dfrac{\omega}{\sigma_i} \ln\left(\dfrac{\lambda_i^{=}(\bar{t})}{\lambda_i^{<}(\bar{t})}\right), & \bar{t} = \bar{T}. \end{cases} \tag{11}$$

By definition, $\mu_i^{=} > 0$.

Below we omit bars for the sake of simplicity.

If the j-th bank defaults at $\tau_j < T$, then for the i-th bank the default boundary is given by

$$\tilde{\mu}_{ij}(t) = \begin{cases} \tilde{\mu}_{ij}^{<} \equiv \dfrac{\omega}{\sigma_i} \ln\left(\dfrac{\tilde{\lambda}_{ij}^{<}(t)}{\lambda_{ij}^{<}(t)}\right), & t < T, \\ \tilde{\mu}_{ij}^{=} \equiv \dfrac{\omega}{\sigma_i} \ln\left(\dfrac{\tilde{\lambda}_{ij}^{=}(t)}{\lambda_{ij}^{<}(t)}\right), & t = T. \end{cases} \tag{12}$$

Note, that according to Eq. (2) $\tilde{\mu}_{ij}^{<}$ doesn't depend on t.

It can be seen that the boundary condition in Eq. (12) at $t = T$ doesn't match to the terminal condition which, according to Eq. (8), reads

$$\tilde{\mu}_{ij,T} = \frac{\omega}{\sigma_i} \ln\left(\frac{\tilde{\lambda}_{ij}(T)}{\lambda_{ij}^{<}(T)}\right) \ne \tilde{\mu}_{ij}^{=}(T). \tag{13}$$

Mathematically, this means that our problem belongs to the class of problems with a boundary (transition) layer at $t = T$. Financially, the behavior of the solution in this layer is determined by the detailed specification of the contract. For instance, if the bank is close to maturity, say a day before, the recovery rate could be defined to smoothly transit from R_i to 1 within this last day. Or, some other conditions specific to the contract in question could be issued. However, we don't consider these details, assuming that the boundary layer is thin, and, therefore, any perturbation of the solution due to the existence of this layer, is dumped out pretty fast when moving away from this layer. In other words, as we ignore a detailed consideration of the boundary layer, our solution experiences a jump at $t = T$. Therefore, after the closed form solution is found we will compare it with the numerical solution of this problem to reveal sensitivity of the former to the value of the described effect.

Below we provide all the results just for two-dimensional case $N = 2$ while the multi-dimensional case will be presented elsewhere. Accordingly, in the definition of $\tilde{\mu}_{ij}^{=}$, $\tilde{\mu}_{ij}^{<}$ for easiness of reading we will omit the second index as in this case it doesn't cause any confusion.

3 Governing equations

Based on the analysis presented in the previous section, the joint survival probability $Q(t, X_1, X_2)$ of two assets X_1, X_2 is defined in the domain $\Omega(t, X_1, X_2) : [0, T] \times$

$[0, \infty] \times [0, \infty]$.[4] It solves the following terminal boundary value problem (Lipton and Sepp 2011)

$$
\begin{aligned}
&Q_t(t, X_1, X_2) + \mathcal{L}Q(t, X_1, X_2) = 0, \\
&Q(T, X_1, X_2) = \mathbf{1}_{X \in \mathbf{D}_{12}}, \quad Q(t, 0, X_2) = 0, \quad Q(t, X_1, 0) = 0,
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
&\mathcal{L}Q = \Delta_\rho Q + \xi \cdot \nabla Q, \\
&\Delta_\rho \equiv \frac{1}{2}\frac{\partial^2}{\partial X_1^2} + \rho \frac{\partial^2}{\partial X_1 \partial X_2} + \frac{1}{2}\frac{\partial^2}{\partial X_2^2}, \quad \xi = (\xi_1, \xi_2)^T,
\end{aligned}
$$

$\mathbf{1}_x$ is the Heaviside step function defined with the half-maximum convention,[5] and the area $\mathbf{D}_{12}$ is defined in Fig. 2. We emphasize that the domain $\mathbf{D}_i$ in $\mathbf{X}$ variables has a curvilinear boundary which depends on the value of $A_{\bar{i}}(T)$. Indeed, based on the definitions in Eqs. (3), (8), (9) and (13), one can find, e.g., for $i = 1$

$$
\tilde{\mu}_{1,T} = \frac{\omega}{\sigma_1} \ln\left[\frac{L_1 + L_{12} - \tilde{R}_{2,T}(1)L_{21}}{L_1 + L_{12} - L_{21}}\right] = \frac{\omega}{\sigma_1} \ln\left[1 + \frac{L_{21}(1 - \tilde{R}_{2,T}(1))}{L_1 + L_{12} - L_{21}}\right].
$$

Next we define the corresponding marginal survival probabilities $q_i(t, X_1, X_2)$, $i = 1, 2$, which are functions of *both* X_1 and X_2, also in the domain $\Omega(t, X_1, X_2)$. For brevity we provide all definitions and formulae for the first bank ($i = 1$) while for the second one it can be done by analogy. So $q_1(t, X_1, X_2)$ solves the following terminal boundary value problem:

$$
\begin{aligned}
&q_{1,t}(t, X_1, X_2) + \mathcal{L}q_1(t, X_1, X_2) = 0, \\
&q_1(T, X_1, X_2) = \mathbf{1}_{\mathbf{X} \in [\mathbf{D}_{12} \cup \mathbf{D}_1]}, \\
&q_1(t, 0, X_2) = 0, \quad q_1(t, X_1, 0) \equiv \Xi(t, X_1) = \begin{cases} \chi_{1,0}(t, X_1), & X_1 \geq \tilde{\mu}_1^<, \\ 0, & 0 \leq X_1 < \tilde{\mu}_1^<, \end{cases}, \\
&q_1(t, X_1, X_2 \uparrow \infty) = \chi_{1,\infty}(t, X_1), \quad q_1(t, X_1 \uparrow \infty, X_2) = 1.
\end{aligned} \tag{15}
$$

In Eq. (15) the domain $\mathbf{D}_1$ is defined in Fig. 2. Function $\chi_{1,0}(t, X_1)$ is the 1D survival probability, which solves the following terminal boundary value problem

$$
\begin{aligned}
&\partial_t \chi_{1,0}(t, X_1) + \mathcal{L}_1 \chi_{1,0}(t, X_1) = 0, \\
&\chi_{1,0}(T, X_1) = \mathbf{1}_{X_1 > \tilde{\mu}_1^=}, \quad \chi_{1,0}(t, \tilde{\mu}_1^<) = 0,
\end{aligned} \tag{16}
$$

where

$$
\mathcal{L}_i = \frac{1}{2}\frac{\partial^2}{\partial X_i^2} + \xi_i \frac{\partial}{\partial X_i}.
$$

[4] The space sub-domain of Ω corresponds to the dotted area in Fig. 2.

[5] Since the detailed consideration of the transition layer at $t = T$ is omitted, this condition allows getting the correct value of $\chi_1(T, \tilde{\mu}_1^=)$, see Eq. (22).

Accordingly, function $\chi_{1,\infty}(t, X_1)$ is the 1D survival probability, which solves the following terminal boundary value problem

$$
\begin{aligned}
&\partial_t \chi_{1,\infty}(t, X_1) + \mathcal{L}_1 \chi_{1,\infty}(t, X_1) = 0, \\
&\chi_{1,\infty}(T, X_1) = \mathbf{1}_{X_1 > \mu_1^=}, \quad \chi_{1,\infty}(t, 0) = 0.
\end{aligned} \tag{17}
$$

4 Survival probabilities

We solve Eqs. (14) and (15) by introducing the Green's function $G(t, X_1, X_2 | t', X_1', X_2')$, where X_1', X_2' are the initial values of X_1, X_2 at $t = t'$. Below, where it is not confusing, for brevity we will also use the notation $G(t - t', X_1, X_2)$, thus explicitly exploiting the fact that for our problem the Green's function depends only on $t - t'$, and omitting the second pair of arguments. The Green's function solves the following initial boundary value problem

$$
\begin{aligned}
&G_t(t - t', X_1, X_2) - \mathcal{L}^\dagger G(t - t', X_1, X_2) = 0, \\
&G(0, X_1, X_2) = \delta(X_1 - X_1')\delta(X_2 - X_2'), \\
&G(t - t', 0, X_2) = 0, \quad G(t - t', X_1, 0) = 0,
\end{aligned} \tag{18}
$$

where $\mathcal{L}^\dagger = \Delta_\rho - \xi \cdot \nabla$. A simple calculation yields

$$
(QG)_t + \mathcal{L} G Q - Q \mathcal{L}^\dagger G = 0,
$$

or, explicitly,

$$
(QG)_t + \nabla \cdot \begin{pmatrix} \frac{1}{2}\left(Q_{X_1} G - Q G_{X_1}\right) - \rho Q G_{X_2} + \xi_1 Q G \\ \frac{1}{2}\left(Q_{X_2} G - Q G_{X_2}\right) + \rho Q_{X_1} G + \xi_2 Q G \end{pmatrix} = 0.
$$

The Green's theorem (Kythe 2011) yields

$$
\begin{aligned}
Q\left(t', X_1', X_2'\right) &= \int_0^\infty dX_1 \int_0^\infty dX_2 G\left(T - t', X_1, X_2\right) \\
&= \iint\limits_{(X_1, X_2) \in \mathbf{D}_{12}} G\left(\tau', X_1, X_2\right) dX_1 dX_2,
\end{aligned} \tag{19}
$$

where $\tau' = T - t'$. Similarly,

$$
\begin{aligned}
q_1(t', X_1', X_2') = &\iint\limits_{(X_1, X_2) \in [\mathbf{D}_{12} \cup \mathbf{D}_1]} G\left(\tau', X_1, X_2\right) dX_1 dX_2 \\
&+ \frac{1}{2} \int_{t'}^{\tau'} ds \int\limits_{\tilde{\mu}_1^<}^{\infty} dX_1 G_{X_2}\left(\tau' - s, X_1, 0\right) \chi_{1,0}\left(s, X_1\right) \\
&- \int\limits_0^\infty dX_1 G_{X_2}\left(\tau' - s, X_1, 0\right) \chi_{1,\infty}\left(s, X_1\right)
\end{aligned} \tag{20}
$$

We start with noting that the 1D Green's function $g_1(\vartheta, X_1),\ \vartheta \equiv t - t'$ at $X_2(t) \leq \mu_2^<$ has the form

$$g_1(\vartheta, X_1) = \frac{e^{-\xi_1^2\vartheta/2+\xi_1(X_1-X_1')}}{\sqrt{2\pi\vartheta}}\left[e^{-\frac{(X_1-X_1')^2}{2\vartheta}} - e^{-\frac{(X_1+X_1'-2\tilde{\mu}_1^<)^2}{2\vartheta}}\right]. \tag{21}$$

Accordingly,

$$\begin{aligned}\chi_{1,0}(t', X_1') &= \int_{\tilde{\mu}_1^=}^{\infty} e^{-\xi_1^2\tau'/2+\xi_1(X_1-X_1')}\left[\frac{e^{-\frac{(X_1-X_1')^2}{2\tau'}}}{\sqrt{2\pi\tau'}} - \frac{e^{-\frac{(X_1+X_1'-2\tilde{\mu}_1^<)^2}{2\tau'}}}{\sqrt{2\pi\tau'}}\right]dX_1 \\ &= \int_{\tilde{\mu}_1^=}^{\infty}\frac{e^{-\frac{(X_1-X_1'-\xi_1\tau')^2}{2\tau'}}}{\sqrt{2\pi\tau'}}dX_1 - e^{-2\xi_1(X_1'-\tilde{\mu}_1^<)}\int_{\tilde{\mu}_1^=}^{\infty}\frac{e^{-\frac{(X_1+X_1'-2\tilde{\mu}_1^<-\xi_1\tau')^2}{2\tau'}}}{\sqrt{2\pi\tau'}}dX_1 \\ &= N\left(-\frac{\tilde{\mu}_1^= - X_1' - \xi_1\tau'}{\sqrt{\tau'}}\right) - e^{-2\xi_1(X_1'-\tilde{\mu}_1^<)}N\left(-\frac{\tilde{\mu}_1^= + X_1' - 2\tilde{\mu}_1^< - \xi_1\tau'}{\sqrt{\tau'}}\right),\end{aligned} \tag{22}$$

and

$$\chi_{1,\infty}(t', X_1') = N\left(-\frac{\mu_1^= - X_1' - \xi_1\tau'}{\sqrt{\tau'}}\right) - e^{-2\xi_1 X_1'}N\left(-\frac{\mu_1^= + X_1' - \xi_1\tau'}{\sqrt{\tau'}}\right).$$

The corresponding 2D Green's function has the form (see Lipton 2001; Zhou 2001 and references therein)

$$\begin{aligned}&G(\vartheta, X_1, X_2) \\ &= \frac{2}{\varpi\vartheta\bar{\rho}}e^{-\frac{\langle\xi^T,\theta\rangle\vartheta}{2}+\langle X-X',\theta\rangle-\frac{R^2+R'^2}{2\vartheta}}\sum_{n=1} I_{\nu_n}\left(\frac{RR'}{\vartheta}\right)\sin(\nu_n\phi)\sin\left(\nu_n\phi'\right),\end{aligned} \tag{23}$$

where $\langle, \rangle$ denotes the dot product, $I_k(x)$ is the modified Bessel function of the first kind,

$$C = \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}, \quad C^{-1} = \frac{1}{\bar{\rho}^2}\begin{pmatrix}1 & -\rho\\ -\rho & 1,\end{pmatrix} \quad \theta = C^{-1}\xi, \quad \nu_n = \frac{n\pi}{\varpi}, \quad \bar{\rho}^2 = 1-\rho^2,$$

$$\varpi = \begin{cases}\pi + \arctan(-\bar{\rho}/\rho), & \rho > 0\\ \pi/2, & \rho = 0,\\ \arctan(-\bar{\rho}/\rho), & \rho < 0,\end{cases} \quad R^2 = \langle X, C^{-1}X^T\rangle, \quad R'^2 = \langle X', C^{-1}X'^T\rangle,$$

$$
\Phi(X_1, X_2) = \begin{cases} \pi + \arctan\left(\dfrac{\bar\rho X_1}{-\rho X_1 + X_2}\right), & X_2 < \rho X_1, \\ \pi/2, & X_2 = \rho X_1, \\ \arctan\left(\dfrac{\bar\rho X_1}{-\rho X_1 + X_2}\right), & X_2 > \rho X_1, \end{cases}
$$
$$
\phi = \Phi(X_1, X_2), \quad \phi' = \Phi(X_1', X_2'), \quad X = (X_1, X_2).
$$

Accordingly,

$$
G_{X_2}(\vartheta, X_1, 0) = \frac{2}{\varpi \vartheta X_1} e^{-\frac{\langle \xi^T, \theta\rangle \vartheta}{2} + \theta_1 X_1 - \langle X', \theta\rangle - \frac{X_1^2/\bar\rho^2 + R'^2}{2\vartheta}} \times \sum_{n=1} (-1)^{n+1} \nu_n I_{\nu_n} \left(\frac{X_1 R'}{\bar\rho \vartheta}\right) \sin\left(\nu_n \phi'\right). \tag{24}
$$

Substitution of these formulas into Eqs. (19) and (20) yields semi-analytical expressions for Q and q_1. However, from the computational point of view, it is more efficient to introduce a new function

$$
\bar q_1(t, X_1, X_2) = q_i(t, X_1, X_2) - \chi_{1,\infty}(t, X_1).
$$

In contrast to $q_1(t, X_1, X_2)$ this new function solves a problem similar the problem given in Eq. (15), but with a homogeneous upper boundary condition:

$$
\begin{aligned}
&\bar q_{1,t}(t, X_1, X_2) + \mathcal{L}\bar q_1(t, X_1, X_2) = 0, \\
&\bar q_1(t, 0, X_2) = \bar q_1(t, X_1, X_2 \uparrow \infty) = 0, \\
&\bar q_1(T, X_1, X_2) = -\mathbf{1}_{X \in \bar{\mathbf{D}}},
\end{aligned} \tag{25}
$$

where $\bar{\mathbf{D}}$ is the area inside the curvilinear triangle with the vertexes in points 6–7–9 in Fig. 2.

As the equations in Eqs. (15) and (25) differ just by the source term, the Green's function of Eq. (25) is also given by Eq. (23). Accordingly, the solution of Eq. (25) reads

$$
\begin{aligned}
q_1(t', X_1', X_2') = \chi_{1,\infty}(t', X_1') - \iint_{(X_1, X_2) \in \bar{\mathbf{D}}} G\left(\tau', X_1, X_2\right) dX_1 dX_2 \\
+ \frac{1}{2}\int_0^{\tau'} ds \int_0^\infty G_{X_2}(\tau' - s, X_1, 0) \left[\Xi(\tau' - s, X_1) - \chi_{1,\infty}(\tau' - s, X_1)\right] dX_1.
\end{aligned} \tag{26}
$$

Another simplification could be made if one wishes to compute the first integral in Eq. (20). For better accuracy it is reasonable to represent it as a difference of two integrals. The first one is taken over the positive quadrant $(X_1, X_2) \in [0, \infty) \times [0, \infty)$

Table 1 Parameters of the structural default model

$L_{1,0}$	$L_{2,0}$	$L_{12,0}$	$L_{21,0}$	R_1	R_2	T	σ_1	σ_2	ρ
60	70	10	15	0.4	0.45	1	1	1	0.5

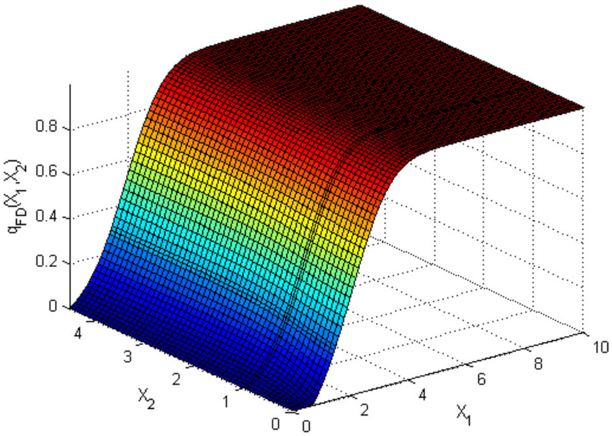

Fig. 3 The marginal survival probability $q_1(X_1, X_2)$ computed by using a Hundsdorfer–Verwer scheme

while the second integral is defined in a union of two semi-infinite strips: $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}$. The idea is that the second integral is defined in the area which is finite either in one or the other direction, and the first integral could be represented in closed form. Therefore, the total computational error is less. We underline that, to the best of our knowledge, representation of the first integral in closed form was not yet given in the literature, so we present this derivation in Appendix 2.

4.1 Numerical experiments

In our test examples we solved Eq. (15) by using first a finite difference scheme (FD) and then compared it with the analytical solution given by Eq. (26). Since Eq. (15) is a pure convection–diffusion two-dimensional problem, we solved it numerically by using a Hundsdorfer–Verwer scheme, see Hout (2010). A non-uniform finite-difference grid was constructed similar to Itkin and Carr (2011) with the grid nodes concentrated close to $\tilde{\mu}_i^{=}, \ i = 1, 2$. We solved the problem using parameters given in Table 1[6]:

We computed all tests using a 100×100 spatial grid. Also we used a constant step in time $\Delta\tau = 0.01$, so that the total number of time steps for a given maturity is 100. The marginal survival probability $q_1(X_1, X_2)$ at $t = 0$ computed by using this method is presented in Fig. 3.

It is easy to see that for our chosen parameters $\tilde{\mu}_1^{<} = 0.6659, \ \tilde{\mu}_2^{<} = 0.2548, \ \mu_1^{=} = 1.4424, \ \mu_2^{=} = 0.9764, \ \tilde{\mu}_1^{=} = 1.5821, \ \tilde{\mu}_2^{=} = 1.0534$.

[6] In our setting the value of the interest rate r doesn't matter.

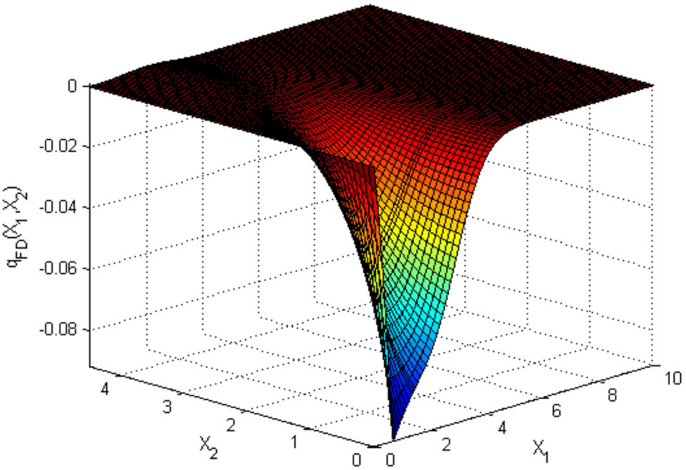

Fig. 4 The difference between marginal survival probabilities $q_1(X_1, X_2)$ computed with and without mutual obligations using the FD method

To observe the effect of the mutual liabilities we repeated this test, but with zero mutual liabilities. Therefore, as compared with the previous case, now the total assets of the i-th bank are $A_i + \sum_j L_{ij}$ while its liabilities are $L_i + \sum_j L_{ji}$. But to provide a correct comparison we need to keep the asset values A_i constant. Therefore, in this case we re-adjust liabilities to $L_i + \sum_j L_{ji} - \sum_j L_{ij}$. In words, this means that, if $\sum_j L_{ij}$ is positive, the bank i gets extra cash and then spends it retiring some of its external liabilities. If this amount is negative, then it is borrowed from the external sources. After this adjustment is done we set $L_{21} = L_{12} = 0$ in our calculations. In what follows, we call this procedure an Adjustment Procedure (AP).

The difference of two solutions is presented in Fig. 4. The above plot clearly demonstrates a significant difference in the solution in the area close to $X_1 = \mu_1^{=}, \; X_2 = \mu_2^{=}$, i.e. the effect of the mutual liabilities is pronounced in this area.

Next we want to compare the analytical and FD solutions. Since the integrands in Eq. (26) are highly oscillating functions, to get a reasonable accuracy we used a Gauss–Kronrod algorithm in both directions. Figure 5 demonstrates the difference in these solutions when there are no mutual obligations. Both solutions coincide pretty well. However, when mutual obligations are taken into account the difference increases as it can be seen in Fig. 6. The difference is bigger in the area closer to $X_1 = \mu_1^{=}$.

Performance-wise, computation of the marginal probabilities on this spatial grid using the FD scheme takes 21 s for $T = 1$ year. At the same time, computation of a single point on a grid using our analytical methods takes 0.4–0.6 s.[7] Therefore, if the marginal probabilities should be computed at every node of the FD grid, using the analytical method it would take about 4000 s, which is pretty slow. However, when calibrating the model with unknown volatilities and the correlation coefficient, we need functions at only three quotes that could be taken from the market values of

[7] We ran this test in Matlab on a standard PC with Intel Xeon E5620 2.4 Ghz CPU.

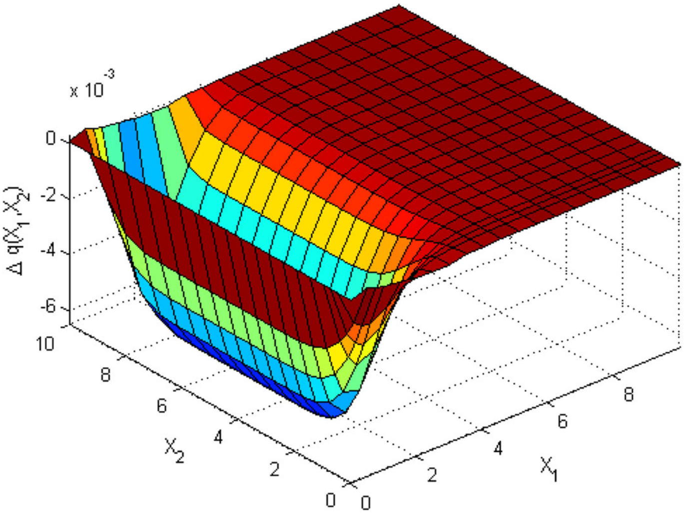

Fig. 5 The difference between marginal survival probabilities $q_1(X_1, X_2)$ computed by the analytical and FD methods with no mutual obligations

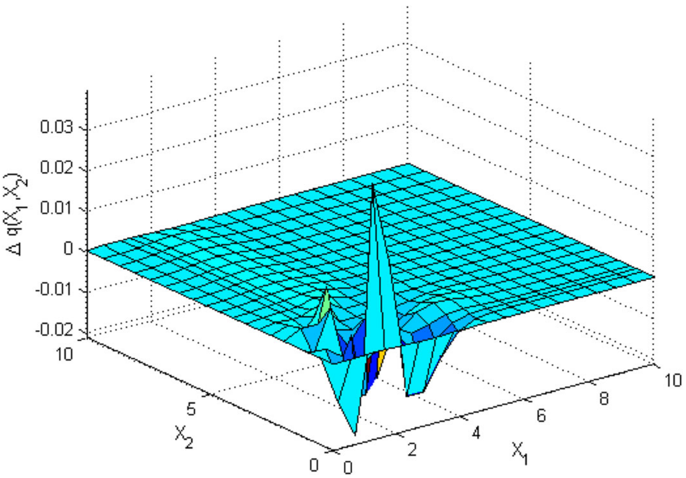

Fig. 6 The difference between marginal survival probabilities $q_1(X_1, X_2)$ computed by the analytical and FD methods with mutual obligations, $T = 5$ years

the CDS spreads and First-To-Default swaps spreads. Thus, in our simplistic model we need just three points, which takes about 1.2 s to compute using our analytics. In contrast, the FD scheme cannot be reduced just to three points on the grid, and, therefore, for such kind of calibration is much less efficient than the analytical method. This is the reason we propose the approach of this paper for doing fast calibration of the model.

In a more general setting, e.g., the one proposed in Itkin and Lipton (2014), this simplified approach could be used to produce a "smart" initial guess for the parameters of the marginal distributions. Then, using this guess, the whole rather complicated problem could be calibrated much faster than starting with some arbitrary values of the parameters, since in this case only relatively small increments of the initial guess should be found.

5 Pricing CDS contracts

We now describe how to price CDSs and FTD in our setting.

5.1 CDS

The price of a CDS $C_1(t, X_1, X_2)$[8] written on the first bank solves the following problem (Bielecki and Rutkowski 2004):

$$\begin{aligned}
&C_{1,t}(t, X_1, X_2) + \mathcal{L}C_1(t, X_1, X_2) = \varsigma_1, \\
&C_1(t, X_1, 0) = \Psi(t, X_1) = \begin{cases} c_{1,0}(t, X_1), & X_1 > \tilde{\mu}_1^< \\ 1 - R_1, & X_1 \le \tilde{\mu}_1^< \end{cases}, \\
&C_1(t, 0, X_2) = 1 - R_1, \quad C_1(t, X_1, X_2 \uparrow \infty) = c_{1,\infty}(t, X_1),
\end{aligned} \tag{27}$$

where ς_i is the coupon rate, $c_{1,0}(t, X_1)$ is the solution of the one-dimensional terminal boundary value problem

$$\begin{aligned}
&\partial_t c_{1,0}(t, X_1) + \mathcal{L}_1 c_{1,0}(t, X_1) = \varsigma_1, \\
&c_{1,0}(t, \tilde{\mu}_1^<) = 1 - R_1, \quad c_{1,0}(t, \infty) = -\varsigma_1(T - t), \\
&c_{1,0}(T, X_1) = (1 - R_1)\mathbf{1}_{\tilde{\mu}_1^< \le X_1 \le \tilde{\mu}_1^=},
\end{aligned} \tag{28}$$

and $c_{1,\infty}(t, X_1)$ is the solution of another one-dimensional terminal boundary value problem

$$\begin{aligned}
&\partial_t c_{1,\infty}(t, X_1) + \mathcal{L}_1 c_{1,\infty}(t, X_1) = \varsigma_1, \\
&c_{1,\infty}(t, 0) = 1 - R_1, \quad c_{1,\infty}(t, \infty) = -\varsigma_1(T - t), \\
&c_{1,\infty}(T, X_1) = (1 - R_1)\mathbf{1}_{X_1 \le \mu_1^=}.
\end{aligned} \tag{29}$$

Also, the statement of problem given in Eq. (27) must be supplied with the terminal condition $C_1(T, X_1, X_2)$, which could be provided based on the picture presented in Fig. 2. Omitting some intermediate algebra, we obtain the following condition

$$\begin{aligned}
&C_1(T, X_1, X_2) = \alpha_1 \mathbf{1}_{(X_1, X_2) \in [\hat{\mathbf{D}} \cup \mathbf{D}_2]} \\
&\alpha_1(X_1, X_2) = \begin{cases} 1 - \min[\tilde{R}_{1,T}(1), R_1], & (X_1, X_2) \in \mathbf{D}_2 \\ 1 - \min[\tilde{R}_{1,T}(\gamma_2), R_1], & (X_1, X_2) \in \hat{\mathbf{D}}. \end{cases}
\end{aligned}$$

[8] For $C_2(t, X_1, X_2)$ similar expressions could be provided by analogy.

The value of the components γ_i at $(X_1, X_2) \in \hat{\mathbf{D}}$ are determined by solving the detailed balance equations which follow from the general N-dimensional problem given in Eq. (6)

$$A_1 + \gamma_2 L_{21} = \gamma_1 (L_1 + L_{12}),$$
$$A_2 + \gamma_1 L_{12} = \gamma_2 (L_2 + L_{21}).$$

The solution in the original variables reads

$$\gamma_i = \frac{L_{\bar{i}} A_i + L_{\bar{i},i}(A_i + A_{\bar{i}})}{\triangle}, \quad i = 1, 2.$$

Observe that the Green's function for Eq. (29) is that given by Eq. (21). Therefore,

$$c_{1,0}(t', X_1') = (1-R_1)\int_{\tilde{\mu}_1^<}^{\tilde{\mu}_1^=} g_1(\tau', X_1) dX_1 + \frac{1-R_1}{2}\int_0^{\tau'} \left.\frac{\partial g_1(\tau'-s, X_1)}{\partial X_1}\right|_{X_1=\tilde{\mu}_1^<} ds$$
$$- \varsigma_1 \int_0^{\tau'}\int_{\tilde{\mu}_1^<}^{\infty} g_1(\tau'-s, X_1) dX_1 ds \equiv I_1 + I_2 + I_3. \quad (30)$$

All these integrals can be computed in closed form. Omitting some intermediate algebra we provide just the final results:

$$I_1 = (1-R_1)\left\{ e^{-2\xi_1(X_1' - \tilde{\mu}_1^<)}\left[N(-y_-) - N(-2y_- - z)\right] + N(y_+) - N(z)\right\},$$
$$I_2 = (1-R_1)\left[e^{-2\xi_1(X_1' - \tilde{\mu}_1^<)} N(y_-) + N(-y_+)\right],$$
$$I_3 = -\varsigma_1 \tau' \left[1 - \frac{y_+}{\xi_1\sqrt{\tau'}} N(-y_+) - e^{-2\xi_1(X_1' - \tilde{\mu}_1^<)} \frac{y_-}{\xi_1\sqrt{\tau'}} N(y_-)\right],$$
$$y_\pm = \frac{\pm(X_1' - \tilde{\mu}_1^<) + \xi_1\tau'}{\sqrt{\tau'}}, \quad z = \frac{X_1' - \tilde{\mu}_1^= + \tau'\xi_1}{\sqrt{\tau'}}. \quad (31)$$

By analogy

$$c_{1,\infty}(t', X_1') = (1-R_1)\int_0^{\mu_1^=} \bar{g}_1(\tau', X_1) dX_1 + \frac{1-R_1}{2}\int_0^{\tau'} \left.\frac{\partial \bar{g}_1(\tau'-s, X_1)}{\partial X_1}\right|_{X_1=0} ds$$
$$- \varsigma_1 \int_0^{\tau'}\int_0^{\infty} \bar{g}_1(\tau'-s, X_1) dX_1 ds, \quad (32)$$

where $\bar{g}_1(\tau', X_1)$ can be obtained from $g_1(\tau', X_1)$ by setting in Eq. (21) $\tilde{\mu}_1^< = 0$. Accordingly, these integrals in closed form are given by Eq. (31) with $\tilde{\mu}_1^< = 0$ and $\tilde{\mu}_1^= = \mu_1^=$.

Using the same trick as in the previous section when we computed the marginal survival probability $q_1(t, X_1, X_2)$, the final solution of this problem could be represented as follows:

$$
\begin{aligned}
C_1(t', X_1', X_2') &= c_{1,\infty}(t', X_1') + \int_0^\infty \int_0^\infty \phi(X_1, X_2) G\left(\tau', X_1, X_2\right) dX_1 dX_2 \\
&+ \frac{1}{2}\int_0^{\tau'} ds \int_0^\infty G_{X_2}(\tau' - s, X_1, 0)\left[\Psi(\tau' - s, X_1) - c_{1,\infty}(\tau' - s, X_1)\right] dX_1, \\
\phi(X_1, X_2) &\equiv \alpha_1 \mathbf{1}_{(X_1, X_2) \in [\hat{\mathbf{D}} \cup \mathbf{D}_2]} - (1 - R_1)\mathbf{1}_{X_1 < \mu_1^=},
\end{aligned} \tag{33}
$$

where the Green's function $G\left(\tau', X_1, X_2\right)$ is given in Eq. (23).

5.2 Numerical experiments

It is well-known in a theory of heat conduction that a direct implementation of Eq. (33) is still impractical. The reason is that at $X_2' = 0$ the first integral in Eq. (33) vanishes, so the second one must converge to $c_{1,0}(t', X_1') - c_{1,\infty}(t', X_1')$ to provide the correct boundary condition at $X_2' = 0$. However, as it could be checked, at $X_2' = 0$ we have $G_{X_2}(\tau' - s, X_1, 0) = 0$, and, hence, the formal validation of Eq. (20) fails at the boundary. As it is explained, e.g. in Kartashov (2001), the reason is that the series in the second integral in Eq. (33) is not uniformly convergent at $X_2' = 0$, so the transition to the limit $X_2' \to 0$ using this representation is complicated and impractical from the computational point of view.

This problem, however, can be overcome by applying another elegant trick that we describe in more detail in Appendix 3.

Further on, we ran the same test as in the previous section with parameters of the model given in Table 1, and $\varsigma_1 = 0.02$. We used the same FD method to verify our solution, see the previous section for the description of the method. The CDS prices for $t = 0$ are presented in Fig. 7.

Again, to observe the effect of mutual liabilities we perform an equivalent computation, but with zero mutual liabilities and the AP applied. The difference of two solutions is presented in Fig. 8. As one would expect, our results demonstrate a significant difference in the area close to $X_1 = \mu_1^=,\ X_2 = \mu_2^=$, i.e. the effect of the mutual liabilities is pronounced in this area not only for the marginal survival probabilities, but for CDS prices as well.

6 Pricing first-to-default (FTD) contracts

The price of the FTD $F_{1,t}$ solves the following terminal boundary value problem[9]:

$$
\begin{aligned}
& F_{1,t} + \mathcal{L}_1 F_1 = \varsigma_1, \\
& F_1(t, 0, X_2) = 1 - R_1, \quad F_1(t, X_1, 0) = 1 - R_2. \\
& F_1(t, X_1, X_2 \uparrow \infty) = f_{1,\infty}(t, X_1), \quad F_1(t, X_1 \uparrow \infty, X_2) = f_{2,\infty}(t, X_2), \\
& F_1(T, X_1, X_2) = \beta_0 \mathbf{1}_{(X_1, X_2) \in \hat{\mathbf{D}}} + \beta_1 \mathbf{1}_{(X_1, X_2) \in \mathbf{D}_1} + \beta_2 \mathbf{1}_{(X_1, X_2) \in \mathbf{D}_2}.
\end{aligned} \tag{34}
$$

[9] For $F_{2,t}$ this can be done by analogy.

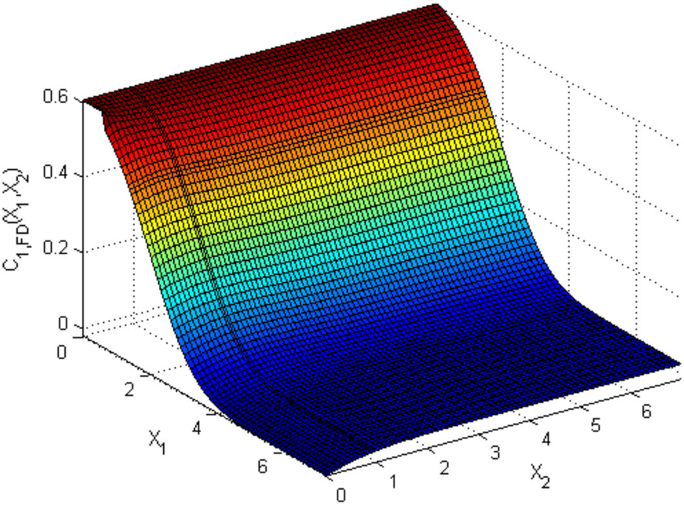

Fig. 7 CDS prices $C_1(X_1, X_2)$ computed by using a Hundsdorfer–Verwer scheme

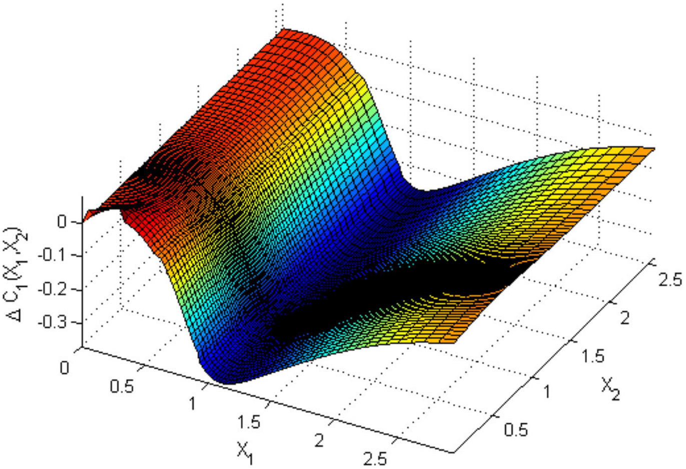

Fig. 8 The difference in CDS prices $C_1(X_1, X_2)$ computed with and without mutual obligations using the FD method

Here $f_{i,\infty}(t, X_i),\ i = 1, 2$ is the solution of the one-dimensional terminal boundary value problem

$$\begin{aligned}
&\partial_t f_{i,\infty}(t, X_i) + \mathcal{L}_i f_{i,\infty}(t, X_i) = \varsigma_i, \\
&f_{i,\infty}(t, 0) = 1 - R_i, \quad f_{i,\infty}(t, \infty) = -\varsigma_i(T - t), \\
&f_{i,\infty}(T, X_i) = (1 - R_i)\mathbf{1}_{X_i \le \mu_i^{=}}.
\end{aligned} \tag{35}$$

As it could be seen, $f_{1,\infty}(t, X_1) = c_{1,\infty}(t, X_1)$ given in Eq. (32). Also $\beta_i = \beta_i(X_1, X_2)$ in Eq. (34) is defined as

$$\beta_i = 1 - \min[\tilde{R}_{\bar{i},T}(1), R_{\bar{i}}], \quad (X_1, X_2) \in \mathbf{D}_i, \ i = 1, 2,$$
$$\beta_0 = 1 - \min[\min[\tilde{R}_{2,T}(\gamma_1), R_2], \min[\tilde{R}_{1,T}(\gamma_2), R_1]], \quad (X_1, X_2) \in \hat{\mathbf{D}}.$$

Similar to the previous section, it can be shown that the solution of this problem reads

$$\begin{aligned} F_1(t', X_1', X_2') = & \int_0^\infty \int_0^\infty \beta_1 \mathbf{1}_{(X_1,X_2)\in[\hat{\mathbf{D}}\cup\mathbf{D}_2]} G\left(\tau', X_1, X_2\right) dX_1 dX_2 \\ & - \varsigma_1 \int_0^{\tau'} \int_0^\infty \bar{g}_1(\tau' - s, X_1) dX_1 ds \\ & + \frac{1}{2}(1 - R_2) \int_0^{\tau'} ds \int_0^\infty G_{X_2}(\tau' - s, X_1, 0) dX_1 \\ & + \frac{1}{2}(1 - R_1) \int_0^{\tau'} ds \int_0^\infty G_{X_1}(\tau' - s, 0, X_2) dX_2, \end{aligned} \tag{36}$$

where the Green's function $G\left(\tau', X_1, X_2\right)$ again is given in Eq. (23).

6.1 Numerical experiments

For the same reason as before a direct implementation of Eq. (36) is impractical from the computational point of view. However, again a similar trick can be applied to significantly improve the accuracy in computation of the boundary integrals. We describe it in more detail in Appendix 4.

Again we ran the same test as in the previous section with parameters of the model given in Table 1, and the same $\varsigma_1 = 0.02$. We used the same FD method to verify our solution, see the previous section for the description of the method. The FTD prices are shown in Fig. 9.

In order to understand the effect of the mutual liabilities on FTD prices, we repeated this test, but with zero mutual liabilities and the AP applied. The difference of two solutions is presented in Fig. 10. The same picture can be obtained by using our analytical approach.

As in the previous cases mutual obligations significantly influence FTD prices especially in the area close to $X_1 = \mu_1^{=}, \ X_2 = \mu_2^{=}$.

We also present the difference in the CDS and FTD prices for the first bank computed with and without mutual obligations and maturity $T = 5$ years. These results are given in Figs. 11 and 12. It is seen that with the increase of maturity the effect of the mutual obligations decreases and in the limit of very long maturities almost disappears.

To illustrate how the terminal distribution of prices looks like in some particular example (which was schematically given in Fig. 2) in Fig. 13 the difference $F_1(T, X_1, X_2) - C_1(T, X_1, X_2)$ is presented as a function of (X_1, X_2). Obviously, $F_1(T, X_1, X_2)$ is positive in $\mathbf{D}_2$ while $C_1(T, X_1, X_2)$ vanishes there (the red box in the right bottom corner of the Figure). And they also differ in a part of the domain $\tilde{\mathbf{D}}$ (the left bottom corner). In other points of the computational domain the values of $F_1(T, X_1, X_2)$ and $C_1(T, X_1, X_2)$ coincide with each other.

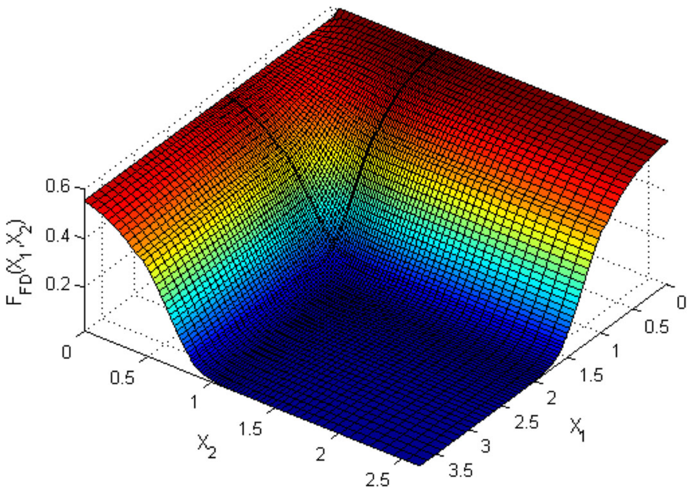

Fig. 9 FTD prices $F_1(X_1, X_2)$ computed by using a Hundsdorfer–Verwer scheme

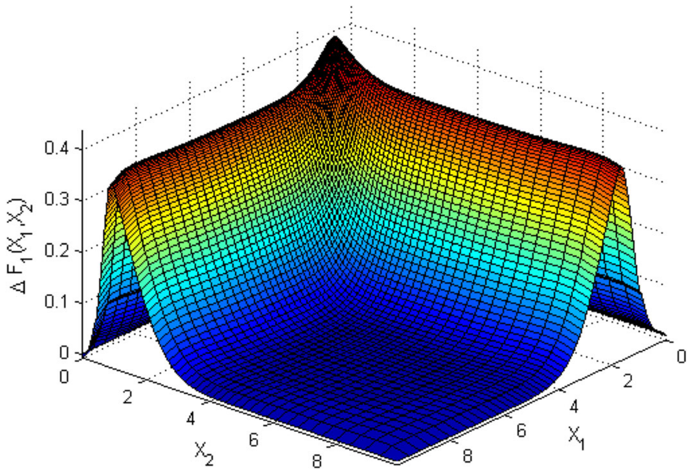

Fig. 10 The difference in FTD prices $F_1(X_1, X_2)$ computed without and with mutual obligations using the FD method

7 Calibration

The model described in Sect. 3 has three unknown parameters: σ_1, σ_2, ρ. We use CDS prices for two assets and the FTD price for both assets to calibrate these parameters. The calibration is done in Matlab using a simple non-linear least square approach where every given point (quote) is taken with the same weight.

In a test experiment we use all the parameters as in Table 1 and also use $A_{1,0} = 300,\ A_{2,0} = 300,\ \varsigma_1 = \varsigma_2 = 0.05$. Then we calibrate σ_1, σ_2, ρ in the following way. First, we set $\sigma_1 = 0.3, \sigma_2 = 0.4, \rho = 0.5$ and compute the prices of CDS and FTD using our algorithm. This gives us the quotes $C_1 = 0.05,\ C_2 = 0.0583,\ F_1 = 0.0583$.

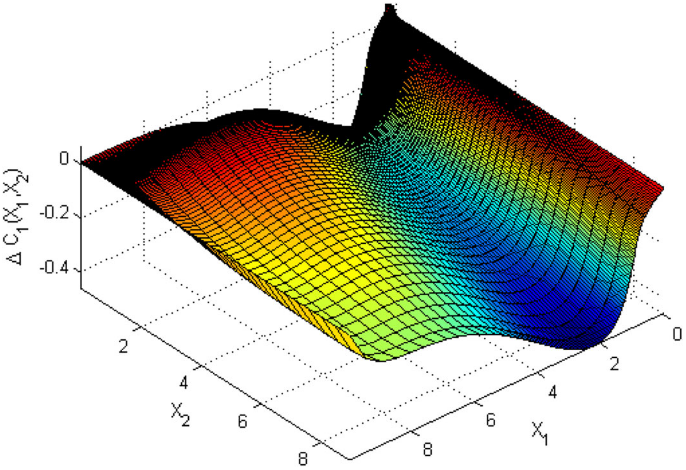

Fig. 11 The difference in CDS prices $C_1(X_1, X_2)$ computed with and without mutual obligations, T = 5 years

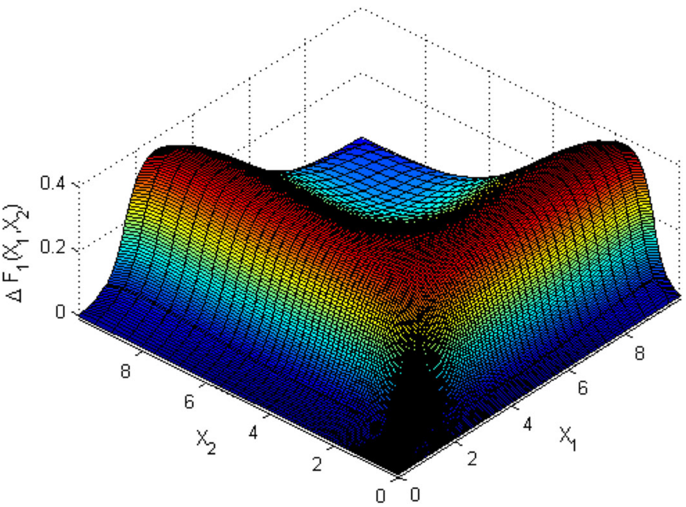

Fig. 12 The difference in FTD prices $C_1(X_1, X_2)$ computed without and with mutual obligations, T = 5 years

Then we run the calibrator to make sure it converges to the same values of σ_1, σ_2, ρ to validate self-consistence of our approach.

Finally, since we investigate how strong the effect of mutual obligations on the parameters of the model is, in the second test we ignore mutual obligations and apply our AP as was discussed in the previous sections. The results of such a calibration are presented in Table 2.

A typical time necessary to get the values of the parameters in Matlab is about 10 s with the objective function tolerance set to 10^{-4}. The corresponding time if the

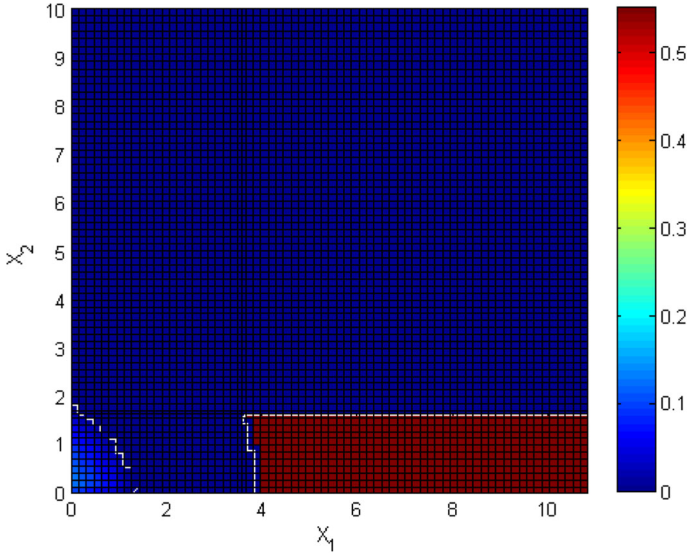

Fig. 13 The difference $F_1(T, X_1, X_2) - C_1(T, X_1, X_2)$, T = 1 year (Color figure online)

Table 2 Results of calibration

Test	T = 1 year			T = 5 years		
	σ_1	σ_2	ρ	σ_1	σ_2	ρ
MO	0.300	0.400	0.500	0.300	0.400	0.500
NMO	0.2819	0.4421	0.4936	0.3189	0.4234	0.2942
Dif (%)	6.0372	−10.5373	1.2801	−6.3108	−5.8616	41.1670

FD algorithm is used with the grid 70 × 70 points in space and time step 0.03 is about 12 times slower. Certainly, for longer maturities this difference increases. The results for $T = 5$ years are also presented in Table 2. Here the computed quotes are $C_1 = 0.2579,\ C_2 = 0.3182,\ F_1 = 0.336$. As can be seen, accounting for the mutual obligations significantly affects the values of the calibrated parameters.

8 Conclusion

In this paper we consider interlinkage (mutual obligations) of banks and their influence on marginal survival probabilities as well as CDS and FTD prices of the corresponding names. We use a simple model where banks' assets are driven by the correlated Brownian motions with drift. The choice of the model is dictated by the advantage to get all the results in closed form, at least in the 2D case. A more sophisticated model with assets driven by some general correlated Lévy processes has been already considered in Itkin and Lipton (2014). However, the present description is more transparent and allows one to better understand the nature of the effect, and also adds CDS and FTD prices to the picture. In the 2D case we also calibrated this model to some artificial market quotes and showed that the mutual obligations must be taken into account to get the correct values of the model parameters as they significantly influence the results of calibration. To the best of our knowledge these results are new.

Another less important, but perhaps interesting, result is a closed form solution for the marginal survival probabilities for two assets driven by the correlated Brownian motions with drift. This solution was not yet given in the literature, so we present it in this paper. To understand a financial meaning of this solution, we underline that for big banks, due to various regulators requirements, their assets cannot drop down much below their liabilities, which means that their recovery rates R should be almost 1 (or, possibly, even exceed 1). In this case when computing, e.g., marginal survival probabilities, the domain $\mathbf{D}_{12}$ in Fig. 2 becomes a positive octant.

Acknowledgements We thank Darrel Duffie, Dilip Madan, Tore Opsahl and Trina Bills for useful comments. We assume full responsibility for any remaining errors.

Appendix 1: Solution of Eq. (7)

We need to prove that Eq. (7) has a unique solution. The below discussion is an alternative to the solution of this problem given by Eisenberg and Noe (2001).

First, consider two extreme cases. If no banks default, then $a_i + \sum_{j \neq i} \gamma_j l_{ji} \geq 1, \ \forall i \in [1, N]$. Obviously, the solution of Eq. (7) is $\gamma_i = 1, \ \forall i \in [1, N]$.

If all banks default, then Eq. (7) transforms to the form

$$F(\boldsymbol{\gamma}) = \boldsymbol{\gamma},$$

where $F(\boldsymbol{\gamma})$ denotes the lhs of Eq. (7). This is a fixed point problem[10] that can be solved by the fixed point iterations method. A sufficient condition for local linear convergence of fixed point iterations is that the Jacobian $J(F(\boldsymbol{\gamma}))$ has to obey the condition

$$|J(F(\boldsymbol{\gamma}))| < 1. \tag{37}$$

To prove this in our case, represent the Jacobian in the explicit form

$$J(F(\boldsymbol{\gamma})) = \begin{vmatrix} 0 & l_{21} & l_{31} & \dots & l_{N1} \\ l_{12} & 0 & l_{32} & \dots & l_{N2} \\ \dots & \dots & \dots & \dots & \dots \\ l_{1N} & l_{2N} & l_{3N} & \dots & 0 \end{vmatrix} = \left(\prod_{k=1}^{N} \tilde{L}_k \right)^{-1} \begin{vmatrix} 0 & L_{21} & L_{31} & \dots & L_{N1} \\ L_{12} & 0 & L_{32} & \dots & L_{N2} \\ \dots & \dots & \dots & \dots & \dots \\ L_{1N} & L_{2N} & L_{3N} & \dots & 0 \end{vmatrix}$$

By definition for any matrix $|M| = ||m_{ij}||, \ i, j \in [1, N]$

$$det(|M|) = \sum_{\chi \in S_N} \operatorname{sgn}(\chi) \prod_{i=1}^{N} m_{i, \chi_i},$$

[10] This actually is a linear system of equations. However, we want to solve it using a fixed-point iterations method to later apply this technique to the general Eq. (7).

where the sum is computed over all permutations χ of the set $S_N = [1, 2, \ldots, N]$, see Bellman (1960). Since all $L_{ij} \geq 0$ we have

$$J(F(\boldsymbol{\gamma})) < \left[\sum_{\chi \in S_N} \prod_{i=1}^{N} L_{i,\chi_i} \right] \left[\prod_{k=1}^{N} \tilde{L}_k \right]^{-1}. \tag{38}$$

Now observe that the numerator in Eq. (38) is a sum of the products of the N elements, and each such a product (1) is positive, and (2) has its vis-à-vis in the denominator. However, the denominator contains also some additional positive terms, for instance $\prod_{i=1}^{N} L_i$, and therefore, $J(F(\boldsymbol{\gamma})) < 1$.

As an example, when $N = 2$

$$|J(F(\boldsymbol{\gamma}))| = \frac{L_{21}}{L_1 + L_{12}} \frac{L_{12}}{L_2 + L_{21}} < 1.$$

Thus, we proved that the condition Eq. (37) is always satisfied. Therefore, by the Banach fixed-point theorem (Granas and Dugundji 2003) the map $F(\boldsymbol{\gamma}) \to \boldsymbol{\gamma}$ is a contraction mapping on $\boldsymbol{\gamma}$, and this implies the existence and uniqueness of the fixed point since a unit cube where $\boldsymbol{\gamma}$ is defined is a compact metric space.

These two extreme cases naturally give rise to the idea of how to solve Eq. (7) in general by using a fixed-point iteration method. Given the vector $\boldsymbol{\gamma}$ from the previous iteration, we check the condition $a_i + \sum_{j \neq i} \gamma_j l_{ji} < 1$ for all $i \in [1, N]$. If for some $i = k$ this condition is not satisfied, we put $\gamma_k = 1$ and exclude the equation for γ_k from Eq. (7). Otherwise, this equation remains in the system. After this step is completed, and for instance, M out of N variables γ were set to 1, we solve Eq. (7) for the remaining $N - M$ variables. The uniqueness of the solution and convergence of the fixed-point iterations follow from the above proof.

Appendix 2: Closed form representation of the integral

Due to various regulators requirements, assets of large banks cannot drop below their liabilities, which means that their recovery rates R should be close to 1. In this case when computing, e.g., marginal survival probabilities, the domain $\mathbf{D}_{12}$ in Fig. 2 becomes a positive quadrant. Finding an analytical solution for survival probability in a positive quadrant with non-zero drift is a long standing problem, that to the best of the authors' knowledge was not solved yet. A relevant literature includes Lipton (2001), Zhou (2001), Metzler (2010), Blanchet-Scalliet and Patras (2008) and references therein.

From a technical prospective we want to compute the integral

$$Q_1(t', X_1', X_2') = \int_0^{\infty} dX_1 \int_0^{\infty} dX_2 G(\tau', X_1, X_2) \tag{39}$$

where the corresponding 2D Green's function is given by Eq. (23). The closed form solution for this integral is known when the drift ξ in Eq. (39) vanishes, see Metzler

(2010) and references therein. However, if $\xi \neq 0$ the closed form solution is not known yet. Here we derive this representation in the form of series of generalized and confluent hypergeometric functions.

First, using polar coordinates R, ϕ we rewrite Eq. (39) in the form

$$Q_1(t', R', \phi') = \frac{2}{\varpi\vartheta} e^{\kappa} \sum_{n=1}^{\infty} \sin\left(\nu_n \phi'\right) \int_0^{\varpi} \sin(\nu_n \phi) d\phi \int_0^{\infty} R e^{\gamma(\phi) R} e^{-\alpha R^2} I_{\nu_n}(\beta R) dR \tag{40}$$

where

$$\kappa = -\frac{\langle \xi^T, \theta \rangle \vartheta}{2} - \frac{R'^2}{2\vartheta} - \langle X', \theta \rangle, \quad \beta = R'/\vartheta, \quad \gamma(\phi) = (\theta_2 + \rho\theta_1)\sin\phi + \bar{\rho}\theta_2 \cos\phi. \quad \alpha = \frac{1}{2\vartheta}.$$

Next, we use the Gegenbauer expansion of the complex exponential of two variables in terms of the ultra-spherical (Gegenbauer) polynomials (Abreu 2008)

$$e^{ixs} = \Gamma(\nu) \left(\frac{s}{2}\right)^{-\nu} \sum_{k=0}^{\infty} i^k (\nu + k) J_{\nu+k}(s) C_k^{\nu}(x), \tag{41}$$

where $C_k^{\nu}(x)$ are the Gegenbauer polynomials (Abramowitz and Stegun 1964), and the parameter ν can be arbitrarily chosen. It can also be seen as a Neumann series (Watson 1966) of the exponential e^{ixt}. By changing variables $s = iS$ in Eq. (41) the latter can be transformed to

$$e^{-Sx} = \Gamma(\nu) \left(\frac{S}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (-1)^k (\nu + k) I_{\nu+k}(S) C_k^{\nu}(x). \tag{42}$$

Substitution of this representation with $S = \beta R$ and $x = -\gamma(\phi)/\beta$ into Eq. (40) yields

$$Q_1(t', R', \phi') = \frac{2}{\varpi\vartheta} e^{\kappa} \Gamma(\nu) \left(\frac{\beta}{2}\right)^{-\nu} \sum_{n=1}^{\infty} \sum_{\mu=0}^{\infty} (-1)^{\mu} (\nu + \mu) \sin\left(\nu_n \phi'\right) \times \int_0^{\varpi} \sin(\nu_n \phi) C_{\mu}^{\nu}(-\gamma(\phi)/\beta) d\phi \int_0^{\infty} R^{1-\nu} e^{-\alpha R^2} I_{\nu_n}(\beta R) I_{\nu+\mu}(\beta R) dR. \tag{43}$$

For the sake of simplicity, it does make sense to choose $\nu = 1$, and then use the identity (Gradshtein and Ryzhik 2007)

$$\int_0^\infty e^{-\alpha R^2} I_{\nu_n}(\beta R) I_\mu(\beta R) dR = 2^{-\nu_n-\mu-1}\alpha^{-(\nu_n+\mu+1)/2}\beta^{\nu_n+\mu}\frac{\Gamma[(1+\mu+\nu_n)/2]}{\Gamma(\mu+1)\Gamma(\nu_n+1)} \times {}_3F_3\left[\begin{matrix}\frac{\nu_n+\mu+1}{2} & \frac{\nu_n+\mu+2}{2} & \frac{\nu_n+\mu+1}{2} \\ \mu+1 & \nu_n+1 & \mu+\nu_n+1\end{matrix}; \frac{\beta^2}{\alpha}\right], \tag{44}$$

where ${}_3F_3(a_1, a_2, a_3; b_1, b_2, b_3; z)$ is a generalized hypergeometric function (Askey and Daalhuis 2010).

Further, observe that at $\nu = 1$ the Gegenbauer polynomials become the Chebyshev polynomials of the second kind which admit the representation (Abramowitz and Stegun 1964)

$$U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k C_k^{n-k} (2x)^{n-2k},$$

where $[x]$ is the floor function. Therefore, the first integral in Eq. (43) assumes the form

$$I_1 = \sum_{k=0}^{[\mu/2]} 2^{\mu-2k}(-1)^k C_k^{\mu-k} \int_0^\varpi \sin(\nu_n\phi)(\bar{\theta}_1 \sin\phi + \bar{\theta}_2\cos\phi)^{\mu-2k} d\phi, \tag{45}$$

with

$$\bar{\theta}_1 = \frac{\theta_2 + \rho\theta_1}{\beta}, \quad \frac{\bar{\rho}\theta_2}{\beta},$$

and $C_k^{\mu-k}$ be the binomial coefficient.

The integral in the rhs of Eq. (45) can be taken in closed form and reads

$$\begin{aligned}
&\int_0^\varpi \sin(\nu_n\phi)(\bar{\theta}_1 \sin\phi + \bar{\theta}_2\cos\phi)^{\mu-2k} d\phi \\
&= \frac{\omega 2^{2k-\mu-1}}{\omega^2(\mu-2k)^2 - \pi^2 n^2}\Big\{a_1\left[b_1\mathcal{F}_1(n) + b_2\mathcal{F}_1(-n)\right] \\
&\quad + a_2\left[b_1\mathcal{F}_2(n) + b_2\mathcal{F}_2(-n)\right]\Big\} \\
&\mathcal{F}_1(n) = {}_2F_1\left(2k-\mu, k+\frac{1}{2}\left(\frac{\pi n}{\omega}-\mu\right), k+\frac{1}{2}\left(\frac{\pi n}{\omega}-\mu\right)+1, -1+\frac{2\bar{\theta}_1}{\bar{\theta}_1 - i\bar{\theta}_2}\right), \\
&\mathcal{F}_2(n) = {}_2F_1\left(2k-\mu, k+\frac{1}{2}\left(\frac{\pi n}{\omega}-\mu\right), k+\frac{1}{2}\left(\frac{\pi n}{\omega}-\mu\right)+1, \frac{e^{2i\omega}(\bar{\theta}_1 + i\bar{\theta}_2)}{\bar{\theta}_1 - i\bar{\theta}_2}\right), \\
&a_1 = -\bar{\theta}_2^{\mu-2k}\left(-\frac{i\bar{\theta}_2}{\bar{\theta}_1 - i\bar{\theta}_2}\right)^{2k-\mu}, \quad a_2 = e^{-i\pi n}\left(\frac{\sin\omega - i\cos\omega}{\bar{\theta}_1 - i\bar{\theta}_2}\right)^{2k-\mu}, \\
&b_i = (-1)^{i-1}\omega(\mu-2k) + \pi n, \; i = 1, 2,
\end{aligned} \tag{46}$$

where ${}_2F_1(a, b, c, x)$ is confluent hypergeometric function (Abramowitz and Stegun 1964).

Although in Eq. (46) the integral is represented as a function of a complex argument, it could be shown that it is real. For example,

$$\begin{aligned}
&\int_0^{\varpi} \sin(\nu_n\phi)(\bar\theta_1 \sin\phi + \bar\theta_2 \cos\phi)d\phi = \frac{\pi n\omega}{\omega^2 - \pi^2 n^2}\left[(-1)^n(\bar\theta_1 \sin(\omega)\right. \\
&\qquad \left. +\bar\theta_2 \cos(\omega)) - \bar\theta_2\right] \\
&\int_0^{\varpi} \sin(\nu_n\phi)(\bar\theta_1 \sin\phi + \bar\theta_2 \cos\phi)^2 d\phi = \frac{1}{2\pi^3 n^3 - 8\pi n\omega^2}\Big\{ - 4\omega^3(\theta_1^2 + \theta_2^2) \\
&\qquad + 2\pi^2\theta_2^2 n^2\omega + (-1)^n\omega\Big[\pi^2 n^2\left(\left(\theta_1^2 - \theta_2^2\right)\cos(2\omega) - 2\theta_1\theta_2\sin(2\omega)\right) \\
&\qquad - \left(\theta_1^2 + \theta_2^2\right)\left(\pi^2 n^2 - 4\omega^2\right)\Big]\Big\},
\end{aligned}$$

etc.

Appendix 3: Computationally efficient representation of Eq. (33)

First, let us mention that the problem given in Eq. (27) is defined in the semi-infinite domain $X_1 \in [0, \infty)$, $X_2 \in [0, \infty)$. However, for practical purposes, this infinite domain is always truncated by some reasonably large value M_i, $i = 1, 2$. Thus, we consider Eq. (27) with $X_2 \in [0, M_2]$. Strictly speaking, this truncation will change the Green's function representation (Polyanin 2002), however the error should be small when $X_2' \to \infty$, or in other words, it is within the truncation error of changing the upper boundary from ∞ to M_2.

To exactly match the boundary conditions in Eq. (27), we replace $C_1(t, X_, X_2)$ with a new function

$$\tilde C_1(t, X_, X_2) = C_1(t, X_, X_2) - \frac{X_2}{M_2} c_{1,\infty}(t, X_1) - \left(1 - \frac{X_2}{M_2}\right)\Psi(t, X_1). \tag{47}$$

Function $\tilde C_1(t, X_, X_2)$ solves the following problem:

$$\begin{aligned}
&\tilde C_{1,t}(t, X_1, X_2) + \mathcal{L}\tilde C_1(t, X_1, X_2) = \Xi(t, X_1, X_2), \\
&\tilde C_1(t, X_1, 0) = \tilde C_1(t, 0, X_2) = \tilde C_1(t, X_1, M_2) = \tilde C_1(t, \infty, X_2) = 0, \\
&\tilde C_1(T, X_1, X_2) = \alpha_1 \mathbf{1}_{(X_1,X_2)\in[\hat{\mathbf{D}}\cup\mathbf{D}_2]} - \frac{X_2}{M}(1 - R_1)\mathbf{1}_{X_1 \le \mu_1^=} \\
&\qquad - \left(1 - \frac{X_2}{M}\right)(1 - R_1)\mathbf{1}_{X_1 \le \tilde\mu_1^=}.
\end{aligned} \tag{48}$$

The solution of this problem is given by the formula (Polyanin 2002)

$$\tilde{C}_1(t', X_1', X_2') = \int_0^\infty dX_1 \int_0^\infty dX_2 \tilde{C}_1(\tau', X_1, X_2) G(\tau', X_1, X_2) + \int_0^{\tau'} ds \int_0^\infty dX_1 \int_0^\infty dX_2 \Xi(\tau' - s, X_1, X_2) G(\tau' - s, X_1, X_2). \tag{49}$$

At $X_2 \to 0$ and $X_2 \to \infty$ due to the boundary conditions for $C_1(t, X_1, X_2)$ the new function $\tilde{C}_1(t, X_1, X_2)$ vanishes. Also, according to the boundary conditions $c_{1,\infty}(t, X_1) \to -\varsigma_1(T - t)$ at $X_1 \to \infty$ as well as $c_{1,0}(t, X_1)$, and $C_1(t, X_1, X_2)$. Therefore, in this limit, $\tilde{C}_1(t, X_1, X_2)$ vanishes as well. Finally, at $X_1 = 0$ we have $c_{1,\infty}(t, 0) = c_{1,0}(t, 0) = C_1(t, X_1, X_2) = 1 - R_1$. Therefore, in this limit, $\tilde{C}_1(t, X_1, X_2) = 0$. Thus, function $\tilde{C}_1(t, X_1, X_2)$ satisfies the homogeneous boundary conditions.

Now, let us give an exact representation of $\Xi(t, X_1, X_2)$. We need to apply the operator $\partial_t + \mathcal{L}$ to both parts of Eq. (47) and take into account Eq. (27) for $C_1(t, X_1, X_2)$. Omitting a tedious algebra we obtain

$$\begin{aligned}
\Xi(t, X_1, X_2) &= \sum_{i=1}^{4} a_i(t, X_1, X_2), \\
a_1(t, X_1, X_2) &= \delta(X_1 - \tilde{\mu}_1^<)\tilde{a}_1(t, X_2), \quad \tilde{a}_1(t, X_2) \\
&= \frac{\partial c_{1,0}(t, X_1)}{\partial X_1}\bigg|_{X_1 = \tilde{\mu}_1^<} \frac{X_2 - M_2}{M_2} \\
&\equiv d_1(t) X_2 + d_2(t), \\
a_2(t, X_1, X_2) &= \delta'(X_1 - \tilde{\mu}_1^<)\tilde{a}_2(t, X_1, X_2), \quad \tilde{a}_2(t, X_1, X_2) \\
&= \left[c_{1,0}(t, X_1) - c_{1,0}(t, \tilde{\mu}_1^<)\right] \frac{X_2 - M_2}{2M_2}, \\
a_3(t, X_1, X_2) &= \mathbf{1}_{X_1 > \tilde{\mu}_1^<} \tilde{a}_3(t, X_1, X_2), \quad \tilde{a}_3(t, X_1, X_2) \\
&= \frac{1}{M_2}\Big[\xi_2(t)\left(c_{1,0}(t, X_1) - c_{1,0}(t, \tilde{\mu}_1^<)\right) \\
&\quad + (X_2 - M_2)\left(\varsigma_1 - \partial_t c_{1,0}(t, \tilde{\mu}_1^<)\right) + \rho \partial_t c_{1,0}(t, X_1)\Big] \\
&\equiv X_2 b_1(t, X_1) + b_2(t, X_1), \\
a_4(t, X_1, X_2) &= \tilde{a}_4(t, X_1) = -\varsigma_1 + \frac{1}{M_2}\Big[\xi_2(t)\left(c_{1,\infty}(t, X_1) - c_{1,0}(t, \tilde{\mu}_1^<)\right) \\
&\quad + \rho\left(\partial_t c_{1,0}(t, X_1) - \partial_t c_{1,\infty}(t, X_1)\right)\Big].
\end{aligned} \tag{50}$$

Further, denote

$$J_i = \int_0^{\tau'} ds \int_0^\infty dX_1 \int_0^\infty dX_2 a_i(\tau' - s, X_1, X_2) G(\tau' - s, X_1, X_2).$$

By using Eqs. (49) and (50) we obtain

$$
\begin{aligned}
&\int_0^{\tau'} ds \int_0^\infty dX_1 \int_0^\infty dX_2 \Xi(\tau' - s, X_1, X_2) G(\tau' - s, X_1, X_2) = \sum_{i=1}^4 J_i, \\
&J_1 = \int_0^{\tau'} ds \int_0^\infty \tilde{a}_1(\tau' - s, X_1, X_2) G(\tau' - s, \tilde{\mu}_1^<, X_2) dX_2 \\
&\quad = \int_0^{\tau'} ds \left[d_2(\tau' - s) Y_1(\tau' - s, \tilde{\mu}_1^<) + d_1(\tau' - s) Y_2(\tau' - s, \tilde{\mu}_1^<) \right], \\
&J_2 = \int_0^{\tau'} ds \int_0^\infty \tilde{a}_1(\tau' - s, \tilde{\mu}_1^<, X_1, X_2) G_{X_1}(\tau' - s, \tilde{\mu}_1^<, X_2) dX_2 \\
&\quad = \int_0^{\tau'} ds \left[d_2(\tau' - s) Z_1(\tau' - s, \tilde{\mu}_1^<) + d_1(\tau' - s) Z_2(\tau' - s, \tilde{\mu}_1^<) \right], \\
&J_3 = \int_0^{\tau'} ds \int_{\tilde{\mu}_1^<}^\infty dX_1 \int_0^\infty dX_2 \tilde{a}_3(\tau' - s, X_1, X_2) G(\tau' - s, X_1, X_2) \\
&\quad = \int_0^{\tau'} ds \Big[\int_{\tilde{\mu}_1^<}^\infty dX_1 b_2(\tau' - s, X_1) Y_1(\tau' - s, X_1) \\
&\qquad + \int_{\tilde{\mu}_1^<}^\infty dX_1 b_1(\tau' - s, X_1) Y_2(\tau' - s, X_1) \Big], \\
&J_4 = \int_0^{\tau'} ds \int_0^\infty dX_1 \tilde{a}_3(\tau' - s, X_1) Y_1(\tau' - s, X_1),
\end{aligned}
$$

where

$$
\begin{aligned}
Y_1(t, X_1) &= \int_0^\infty G(t, X_1, X_2) dX_2, \quad Y_2(t, X_1) = \int_0^\infty X_2 G(t, X_1, X_2) dX_2, \\
Z_1(t, X_1) &= \int_0^\infty G_{X_1}(t, X_1, X_2) dX_2, \quad Z_2(t, X_1) = \int_0^\infty X_2 G_{X_1}(t, X_1, X_2) dX_2.
\end{aligned}
$$

Also we emphasize that a pretty similar approach can be used for computing marginal probabilities.

Appendix 4: Computationally efficient representation of Eq. (36)

By using a similar idea as in the previous Appendix we first truncate the infinite domain $(X_1, X_2) \in [0, \infty) \times [0, \infty)$ to a finite domain $(X_1, X_2) \in [0, M_1] \times [0, M_2]$ and introduce a new function

$$
\begin{aligned}
&\tilde{F}_1(t, X_, X_2) = F_1(t, X_, X_2) - h(t, X_1, X_2) h(t, X_1, X_2) \\
&\quad = \left\{ \left[\frac{X_2}{M_2} f_{1,\infty}(t, X_1) + \left(1 - \frac{X_2}{M_2}\right)(1 - R_2) \right] \mathbf{1}_{X_1} + (1 - R_1)\left(1 - \mathbf{1}_{X_1}\right) \right\} \\
&\quad \mathbf{1}_{M_1 - X_1} - f_{2,\infty}(t, X_2) \left[1 - \mathbf{1}_{M_1 - X_1} \right]. \qquad (51)
\end{aligned}
$$

Function $\tilde{F}_1(t, X_, X_2)$ solves the following problem:

$$
\begin{aligned}
&\tilde{F}_{1,t}(t, X_1, X_2) + \mathcal{L}\tilde{F}_1(t, X_1, X_2) = \Upsilon(t, X_1, X_2),\\
&\tilde{F}_1(t, X_1, 0) = \tilde{F}_1(t, 0, X_2) = \tilde{F}_1(t, X_1, M_2) = \tilde{F}_1(t, M_1, X_2) = 0,\\
&\tilde{F}_1(T, X_1, X_2) = F_1(T, X_1, X_2) - h(T, X_1, X_2),
\end{aligned} \tag{52}
$$

and $f_{i,\infty}(T, X_i) = (1 - R_i)\mathbf{1}_{X_i \le \mu_i^=}$.

The solution of this problem is given by the formula (Polyanin 2002)

$$
\begin{aligned}
\tilde{F}_1(t', X_1', X_2') = &\int_0^\infty dX_1 \int_0^\infty dX_2 \tilde{F}_1(\tau', X_1, X_2) G(\tau', X_1, X_2)\\
&+ \int_0^{\tau'} ds \int_0^\infty dX_1 \int_0^\infty dX_2 \Upsilon(\tau' - s, X_1, X_2) G(\tau' - s, X_1, X_2).
\end{aligned} \tag{53}
$$

In order to compute $\Upsilon(t, X_1, X_2)$, apply the operator $\partial_t + \mathcal{L}$ to both parts of Eq. (51) and take into account Eq. (34) for $F_1(t, X_1, X_2)$. Omitting a tedious algebra we obtain

$$
\begin{aligned}
\Upsilon(t, X_1, X_2) = &-2\varsigma_1 + \frac{1}{2}\left[-1 + R_1 + f_{2,\infty}(t, X_2)\right] \delta'_{X_1}(M_1 - X_1)\\
&- \frac{\delta'_{X_1}(0)}{M_2}\left[M_2(R_1 - R_2) - y(1 - R_2 + f_{1,\infty}(t, X_1))\right]\\
&+ \delta(X_1) b_1(t, X_1, X_2) + \delta(M_1 - X_1) b_2(t, X_1, X_2),
\end{aligned} \tag{54}
$$

where $b_i(t, X_1, X_2)$ are some functions. We omit the explicit form of these functions since the integrals

$$
\begin{aligned}
&\int_0^\infty \delta(X_1) G(\tau' - s, X_1, X_2) b_1(t, X_1, X_2) dX_1 = 0,\\
&\int_0^\infty \delta(M_1 - X_1) G(\tau' - s, X_1, X_2) b_2(t, X_1, X_2) = 0,
\end{aligned}
$$

due to the boundary conditions for the Green's function. Therefore, the final representation for the boundary integral in Eq. (53) reads

$$
\begin{aligned}
&\int_0^{\tau'} ds \int_0^\infty dX_1 \int_0^\infty dX_2 \Upsilon(\tau' - s, X_1, X_2) G(\tau' - s, X_1, X_2) = K_1 + K_2 + K_3,\\
&K_1 = -2\varsigma_1 \int_0^{\tau'} ds \int_0^\infty dX_1 Y_1(\tau' - s, X_1),\\
&K_2 = \frac{1}{2}\int_0^{\tau'} ds \int_0^\infty \left[-f_{2,\infty}(t, X_2) + R_1 - 1\right] G_{X_1}(\tau' - s, M_1, X_2) dX_2,\\
&K_3 = \int_0^{\tau'} \left[(R_1 - R_2) Z_1(\tau' - s, 0) + \frac{R_1 + R_2 - 2}{M_2} Z_2(\tau' - s, 0)\right] ds.
\end{aligned}
$$

At numerical (discrete) realization all K_i, $i \in [1, 3]$ vanish at the boundaries as well as the first integral in Eq. (53), and so does $\tilde{F}_1(t, X_1, X_2)$. Therefore, by definition of $\tilde{F}_1(t, X_1, X_2)$ this preserves the correct boundary conditions for $F_1(t, X_1, X_2)$.

References

Abramowitz, M., & Stegun, I. (1964). *Handbook of mathematical functions*. Mineola: Dover Publications Inc.

Abreu, L. (2008). The reproducing kernel structure arising from a combination of continuous and discrete orthogonal polynomials into fourier systems. arXiv:math/0601190.

Askey, R., & Daalhuis, A. B. O. (2010). Generalized hypergeometric function. In F. Olver, D. Lozier, R. Boisvert, & C. Clark (Eds.), *NIST handbook of mathematical functions*. Cambridge: Cambridge University Press.

Bellman, R. (1960). *Introduction to matrix analysis*. New York: McGraw-Hill.

Bielecki, T. R., & Rutkowski, M. R. (2004). *Credit risk: Modeling, valuation and hedging*. Berlin: Springer.

Black, F., & Cox, J. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, *31*(2), 351–367.

Blanchet-Scalliet, C., & Patras, F. (2008). Counterparty risk valuation for CDS. arXiv:0807.0309.

David, A., & Lehar, A. (2014). Why are banks highly interconnected? doi:10.2139/ssrn.1108870.

Eisenberg, L., & Noe, T. (2001). Systemic risk in financial systems. *Management Science*, *47*, 236–249.

Gradshtein, I., & Ryzhik, I. (2007). *Table of integrals, series, and products*. Amsterdam: Elsevier.

Granas, A., & Dugundji, J. (2003). *Fixed point theory*. New York: Springer.

In't Hout, K. J., & Foulon, S. (2010). ADI finite difference schemes for option pricing in the Heston model with correlation. *International Journal of Numerical Analysis and Modeling*, *7*(2), 303–320.

Itkin, A. (2014). Efficient solution of backward jump-diffusion PIDEs with splitting and matrix exponentials. *Journal of Computational Finance* (2014). arXiv:1304.3159 (Forthcoming).

Itkin, A., & Carr, P. (2011). Jumps without tears: A new splitting technology for barrier options. *International Journal of Numerical Analysis and Modeling*, *8*(4), 667–704.

Itkin, A., & Lipton, A. (2014). Efficient solution of structural default models with correlated jumps and mutual obligations. arXiv:1408.6513.

Kartashov, E. (2001). *Analytical methods in the theory of heat conduction in solids*. Moscow: Vysshaya Shkola.

Kythe, P. (2011). *Green's functions and linear differential equations: Theory, applications, and computation. Applied mathematics and nonlinear science*. London: Chapman & Hall/CRC.

Lipton, A. (2001). *Mathematical methods For foreign exchange: A financial engineer's approach*. Hackensack: World Scientific.

Lipton, A., & Savescu, I. (2014). Pricing credit default swaps with bilateral value adjustments. *Quantitative Finance*, *14*(1), 171–188.

Lipton, A., & Sepp, A. (2011). Credit value adjustment in the extended structural default model. In A. Lipton, A. Rennie (Eds.), *The oxford handbook of credit derivatives* (pp. 406–463). Oxford University.

Merton, R. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, *29*, 449–470.

Metzler, A. (2010). On the first passage problem for correlated Brownian motion. *Statistics and Probability Letters*, *80*, 277–284.

Polyanin, A. (2002). *Handbook of linear partial differential equations for engineers and scientists*. London: Chapman & Hall/CRC.

Watson, G. (1966). *A treatise on the theory of Bessel functions* (2nd ed.). Cambridge, UK: Cambridge University Press.

Webber, L., & Willison, M. (2011). Systemic capital requirements. Technical Report 436, Bank of England. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1945654.

Zhou, C. (2001). The term structure of credit spreads with jump risk. *Journal of Banking and Finance*, *25*, 2015–2040.