Physics and Derivatives: *On Three Important Problems in Mathematical Finance*

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KEY FINDINGS

- We introduce a powerful extension of the classical method of heat potentials designed for solving initial boundary value problems for the heat equation with moving boundaries.
- We demonstrate the versatility of our method by solving several classical problems of financial engineering in a unified fashion.
- In particular, we find the boundary corresponding to the constant default intensity in the structural default model, thus solving in the affirmative a long outstanding problem.

ABSTRACT: *In this article, we use recently developed extension of the classical heat potential method in order to solve three important but seemingly unrelated problems of financial engineering: (A) American put pricing; (B) default boundary determination for the structural default problem; and (C) evaluation of the hitting time probability distribution for the general time-dependent Ornstein–Uhlenbeck process. We show that all three problems boil down to analyzing behavior of a standard Wiener process in a semi-infinite domain with a quasi-square-root boundary.*

TOPICS: ***Derivatives, options, credit default swaps****

The method of heat potentials is a highly powerful approach in mathematical physics (Tikhonov and Samarskii 1963, Rubinstein 1971, Landis 1997, Kartashov 2001, Watson 2012). It has been used for decades in several important fields including heat transfer, the Stefan problem, nuclear engineering, and material science. However, it is not so widely used in mathematical finance. The first use of the method of heat potentials in mathematical finance is described in Lipton (2001), who applied it for pricing barrier options with curvilinear barrier (Section 12.2.3, pp. 462–467). In this article, we consider three classical problems of mathematical finance, and show how natural is the application of the method of heat potentials to them.

First, we consider the problem of pricing an American put option; then we consider the problem of computation of the default boundary in a structural default model. Finally, we consider the first hitting time of an Ornstein–Uhlenbeck process. We show that all three problems boil down to analyzing behavior of a standard Wiener process in a semi-infinite domain with a quasi-square-root boundary, which can be done elegantly by using the method of heat potentials.

*All articles are now categorized by topics and subtopics. **View at PM-Research.com.**

The American Put Option

American options are one of the most traded options for several asset classes. It is a well-known fact that for an American call option without dividends, it is never optimal to exercise it early. Therefore, we concentrate our analysis on American put options, assuming that the risk-free rate is *positive*. We note in passing that in several major economies this rate is actually *negative*, which requires rebuilding the very foundation of mathematical finance—a task which is not pursued here.

In the Black–Scholes framework, no closed-form solution exists for the price of an American put, hence numerical or semi-analytical methods should be used. There are many well-known methods for pricing American options. One of the simplest approaches is to use binomial trees (Cox et al. 1979 and Leisen and Reimer 1996). This is very similar to the binomial model for European options but for each node expected payoff of the option is compared to the payoff one gets if the option is exercised immediately.

Finite-difference methods (Brennan and Schwartz 1978, Forsyth and Vetzal 2002, Reisinger and Witte 2012, Duffy 2013), where on each time step of the scheme an exercise decision is made, are well suited for the continuous time case.

Alternatively, the pricing problem can be formulated as a free boundary problem. The main task is to determine the exact location of the early exercise boundary. Many authors reduce the free boundary problem to an integral equation and solve it numerically (Kim 1990, Kuske and Keller 1998, Hou et al. 2000, Aitsahlia and Lai 2001, Kim et al. 2013, Andersen et al. 2016).

There are numerous analytical/semi-analytical approximations of the exact solution. The main advantage of these approximations is that they are very fast to compute, however they might be inaccurate. As examples of analytical/semi-analytical approximations, we mention Barone-Adesi and Whaley (1987), Ju (1998), Ju and Zhong (1999), Carr (1998), Ostrov and Goodman (2002), Zhu (2006), Zhu and He (2007), and many others.

Finally, the least-square Monte Carlo method (Longstaff and Schwartz 2001, Tsitsiklis and Van Roy 2001, Andersen and Broadie 2004), uses regression to estimate the values of the option from simulated paths and thus determine exercise rule and price of the option. This method is the most flexible and allows to price multidimensional options. However, it is very noisy in comparison to other methods.

In this article, we are interested in the free boundary problem approach. Using the method of heat potentials, we show how to reduce this problem to a system of Volterra integral equations of the second kind, which are easily solved numerically.

The Structural Default Problem

There are two main approaches to credit risk modeling: *reduced-form* models and *structural* models. The idea of the reduced-form approach is not to go deep in economics reasons of the default event, but to assume that the default comes randomly and model the likelihood of the default. Some examples of these models are Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Duffie and Singleton (1997), Duffie and Singleton (1999), and Lando (1998), among others. The full review of this approach can be found in Bielecki and Rutkowski (2013) and Lipton and Rennie (2013). The advantage of this approach is that it allows an easy calibration to the market.

In this article, we are interested in the structural approach. In contrast to the reduced-form framework, the structural framework has an economic interpretation of the default. In a seminal work of Merton (1974), the default event is defined as the event when the assets fall below the liabilities at maturity. Many authors propose various extensions of this basic model, Black and Cox (1976), Kim et al. (1993), Nielsen et al. (1993), Leland (1994), Longstaff and Schwartz (1995), Leland and Toft (1996), Albanese and Chen (2005); a detailed exposition is given in Lipton and Rennie (2013). They consider more complicated forms of the debt and assume that the default event can be monitored continuously. The disadvantage of these models is that they are not able to generate sufficiently high short-term CDS spreads typically observed in the market. The next generation of models do this in different ways, for instance, by incorporating jumps in the asset's dynamics (Zhou 2001, Hilberink and Rogers 2002, Lipton 2002, Lipton et al. 2007, Sepp 2004, Sepp 2006, Cariboni and Schoutens 2007, Feng and Linetsky 2008, Lipton and Sepp 2009), or making initial asset uncertain (Finger et al. 2002).

Another approach is considered by Hyer et al. (1999), Hull and White (2001), Avellaneda and Zhu (2001), who propose to make the default barrier curvilinear. Motivated by the latter approach, we develop a new method for calibration of the default boundary to default probabilities observed in the market. We emphasize that for more than two decades it was not known whether a curvilinear boundary capable of producing nontrivial probability of default for very short times actually exists. In this article, we answer this question affirmatively by showing how to construct it for the all-important case of constant default intensity.

First Time Hitting of an Ornstein–Uhlenbeck Process

Computation of the first hitting time density for an OU process is a long-standing problem in mathematical finance, and mathematical physics more generally. Attempts to solve it analytically have been made for many years, but were unsuccessful, see e.g., Leblanc and Scaillet (1998) and Leblanc et al. (2000). There are several papers that derive the density using the Laplace transform for the case with constant coefficients and flat boundary. The direct Laplace transform can be computed analytically, while for the inverse transform the analytical representation is not available. Ricciardi and Sato (1988) compute the inverse transform numerically, while Alili et al. (2005) and Linetsky (2004) find a semi-analytical representation as an infinite series of special functions, which is not very efficient for practical computations. For financial applications, short- and long-term asymptotics play a very important role. Martin et al. (2019) develop a short- and long-time asymptotic expansion for the OU and other mean-reverting processes.

Recently, Lipton and Kaushansky (2018, 2019) used the method of heat potentials and reduced the problem to a Volterra integral equation of the second kind, which can be easily solved numerically. Once the equation is solved, the hitting density can be found by simple trapezoidal integration. The distinct and important advantage of the method, in comparison to other methods, is that it still works for the case of time-dependent coefficients and time-dependent barrier.

In this article, we overview the method of Lipton and Kaushansky (2018, 2019) and present several illustrative numerical results.

Article Structure

The rest of the article is organized as follows: in the next section, we overview the method of heat potentials for solving initial boundary value problems; then we show how to compute the exercise boundary for an American put. After that, we show how to construct the default boundary for a structural default model and then we show how to compute the first hitting time density of an OU process; in the final section, we conclude.

THE METHOD OF HEAT POTENTIALS

Consider the following initial boundary value problem

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx}, \quad && t > 0, x \geq b(t) \\ u(0,x) &= 0, && x \geq b(0), \\ u(t,b(t)) &= f(t), && t > 0. \end{aligned}$$

Using the method of heat potentials (see Tikhonov and Samarskii 1963 or Lipton 2001), the solution can be written as

$$u(t,x) = \int_0^t \frac{(x-b(t'))\exp\left(-\frac{(x-b(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt', \qquad (1)$$

where $\nu(t)$ is a weight function, which solves the following Volterra integral equation of the second kind[1]

$$\nu(t) + \int_0^t \frac{(b(t)-b(t'))\exp\left(-\frac{(b(t)-b(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt' = f(t).$$

Lipton and Kaushansky (2019) derived a similar expression for $u_x(t,b(t))$[2]

$$\begin{aligned} u_x(t,b(t)) = &-2\left(\frac{1}{\sqrt{2\pi t}} + b'(t)\right)\nu(t) \\ &+ \int_0^t \frac{\left(1-\frac{\Psi(t,t')^2}{(t-t')}\right)\Xi(t,t')\nu(t') - \nu(t)}{\sqrt{2\pi(t-t')^3}}dt', \end{aligned}$$

[1] Note that we cannot just plug $x = b(t)$ to (1) due to singularity of the integral. Derivations of the limit are given in Appendix A.

[2] For completeness we provide derivations in Appendix A.

where

$$\Psi(t,t') = b(t) - b(t'),$$

$$\Xi(t,t') = \begin{cases} \exp\left(-\frac{\Psi(t,t')^2}{2(t-t')}\right), & t > t' \\ 1, & t > t' \end{cases}.$$

We use this result in the following sections.

THE AMERICAN PUT OPTION

Consider a stock with price S_t governed by a geometric Brownian motion

$$\frac{dS_t}{S_t} = r\,dt + \sigma dW_t,$$

with risk-free rate r, which is assumed to be positive, $r > 0$, and volatility σ.

We wish to price an American put option with strike K, paying $(K - S_\tau)_+$ if exercised at time $\tau \in [0, T]$. If not previously exercised, the put value is

$$p(t) = \sup_{\tau \in [t,T]} \mathbb{E}_t[e^{-r(\tau-t)}(K - S_\tau)_+].$$

It is known that the decision to exercise the put option is characterized by a deterministic exercise boundary $S_T^*(t)$ (Lipton 2001)

$$\tau^* = \inf\{t \in [0,T] : S_t \le S_T^*(t)\}.$$

Using the Feynman–Kac formula, we can write a PDE for the American put option (Kim 1990)

$$\begin{aligned} &V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \\ &V(t, S_T^*(t)) = K - S_T^*(t), \\ &V_S(t, S_T^*(t)) = -1, \\ &V(T,S) = 0, \end{aligned}$$

Consider a change of variables

$$x = \ln\left(\frac{S}{K}\right), \;\; \tau = \sigma^2(T-t),$$

$$P = \frac{V}{K}, \;\; b(0) = 0, \;\; k = r/\sigma^2. \tag{2}$$

Since $S_T^*(t)$ depends on the maturity T, it is more convenient to consider $b(\tau) = S_T^*(t)$.

Then, equation for the American put becomes

$$\begin{aligned} P_\tau(\tau,x) &= \frac{1}{2}P_{xx}(\tau,x) \\ &\quad + \left(k - \frac{1}{2}\right)P_x(\tau,x) - kP(\tau,x), \\ P(\tau, b(\tau)) &= 1 - e^{b(\tau)}, \\ P_x(\tau, b(\tau)) &= -e^{b(\tau)}, \\ \lim_{x\to\infty} P(\tau,x) &= 0, \\ P(0,x) &= 0. \end{aligned}$$

Let

$$Q(\tau,x) = \exp(\alpha\tau + \beta x)P(\tau,x),$$

where

$$\alpha = \frac{1}{2}\left(k + \frac{1}{2}\right)^2, \quad \beta = \left(k - \frac{1}{2}\right) \equiv \beta_-, \quad \beta_+ = \left(k + \frac{1}{2}\right).$$

Then,

$$\begin{aligned} Q_\tau &= \frac{1}{2}Q_{xx}, \\ Q(\tau, b(\tau)) &= \exp(\alpha\tau)(\exp(\beta_- b(\tau)) \\ &\quad - \exp(\beta_+ b(\tau))) \equiv \phi(\tau), \\ Q_x(\tau, b(\tau)) &= \exp(\alpha\tau)(\beta_- \exp(\beta_- b(\tau)) \\ &\quad - \beta_+ \exp(\beta_+ b(\tau))) \equiv \chi(\tau), \\ \lim_{x\to\infty} Q(\tau,x) &= 0, \\ Q(0,x) &= 0. \end{aligned}$$

Thus, Q satisfies a standard heat equation in a domain with a moving boundary, which is determined in such a way that two boundary conditions become compatible.

Solution Using the Method of Heat Potentials

Using Theorem 1 from Lipton and Kaushansky (2019), we obtain the integral formulation of the problem

$$\nu(\tau) + \int_0^\tau \frac{\Psi(\tau,\tau')\Xi(\tau,\tau')\nu(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau' = \phi(\tau), \tag{3}$$

$$-2\left(\frac{1}{\sqrt{2\pi\tau}}+b'(\tau)\right)\nu(\tau)$$

$$+\int_0^\tau \frac{\left(1-\frac{\Psi(\tau,\tau')^2}{(\tau-\tau')}\right)\Xi(\tau,\tau')\nu(\tau')-\nu(\tau)}{\sqrt{2\pi(\tau-\tau')^3}}d\tau' = \chi(\tau), \quad (4)$$

where

$$\Psi(\tau,\tau') = b(\tau)-b(\tau'), \qquad \tau \geq \tau',$$

$$\Xi(\tau,\tau') = \begin{cases} \exp\left(-\frac{\Psi(\tau,\tau')^2}{2(\tau-\tau')}\right), & \tau > \tau' \\ 1, & \tau = \tau' \end{cases}.$$

According to Ostrov and Goodman (2002), we can represent the boundary for small τ as follows:

$$b(\tau) = -\sqrt{\tau\left|\ln(8\pi k^2\tau)\right|}. \qquad (5)$$

We use this expression to start our equations and then switch to finding the boundary. We can think of $b(\tau)$ as a typical square-root-like boundary.

Numerical Method

We use a piece-wise linear approximation of the exercise boundary on a time grid $\tau_0 < \tau_1 < \ldots < \tau_n = T$. We need to determine $b_i = b(\tau_i)$ and $\nu_i = \nu(\tau_i)$ for $0 \leq i \leq n$. We choose small time τ_0. Then, the exercise boundary up to $t = \tau_0$ can be determined by (5). Using (3) and (4), we are able to determine $\nu(t)$ for $t \in [0, \tau_0]$. We can rewrite

$$b_i = b_0 + \sum_{j=1}^{i}\gamma_j(\tau_j - \tau_{j-1}).$$

Hence, we aim to determine γ_i for $1 \leq i \leq n$.

It is worth noting that $b(t)$ has a discontinuous derivative at $t = t_i$ and (4) is not well-defined. Therefore, we define

$$b_{i-\frac{1}{2}} = b_{i-1} + \gamma_i(\tau_{i-\frac{1}{2}} - \tau_{i-1}),$$

where $\tau_{i-\frac{1}{2}} = \frac{\tau_{i-1}+\tau_i}{2}$, and evaluate (3) and (4) at $t = \tau_{i-\frac{1}{2}}$.

Assume we have found the $\gamma_1, \ldots, \gamma_{i-1}$ Then, we can write

$$b_i(\gamma_i) = b_{i-1} + \gamma_i(\tau_i - \tau_{i-1}), \qquad (6)$$

and

$$b_{i-\frac{1}{2}}(\gamma_i) = b_{i-1} + \gamma_i(\tau_{i-\frac{1}{2}} - \tau_{i-1}), \qquad (7)$$

Using the last equation, we are able to find $\nu_{i-\frac{1}{2}}$ as a function of γ_i by solving (3) numerically. Numerical methods for solving (3) can be found in Lipton and Kaushansky (2019).

Using $b_{i-\frac{1}{2}}(\gamma_i)$ and $\nu_{i-\frac{1}{2}}(\gamma_i)$, we solve (4) with respect to γ_i. It can be easily computed numerically by evaluating the integral with a square-root singularity and noticing that $b'(\tau_{i-\frac{1}{2}}) = \gamma_i$. To compute the integral, we split it in two parts: $[0, \tau_0]$ and $[\tau_0, \tau_{i-\frac{1}{2}}]$. The former integral is evaluated with a higher-order integration methods, and the latter integral is evaluated using a trapezoidal rule.

Therefore, we apply the following method to find the early exercise boundary $b(\tau)$:

Algorithm 1 Numerical Method for the American Put Option

Require: Choose τ_0 manually.

Define a grid $0 = t_0 < t_1 < \ldots < t_m = \tau_0$. Compute $b(t_i)$ using (5).
Find $\nu(t_0), \nu(t_1), \ldots, \nu(t_m)$ by solving (3).
Define a grid $\tau_0 < \tau_1 < \ldots < \tau_n$.
for $i = 1$: n **do**
    Compute (7) as a function of γ_i.
    Find $\nu_{i-\frac{1}{2}}$ as a function of γ_i by solving (3).
    Find γ_i by solving (4) at $\tau_{i-\frac{1}{2}}$ with respect to γ_i.
    Compute b_i using (6).
end for

Numerical Results

Consider the following parameters in Exhibit 1. After change of variables (2), $k = 0.22$ and $\tilde{T} = \sigma^2 T = 0.45$. We also choose $\tau_0 = 0.01$.

In Exhibits 2 and 3, we compute the exercise boundary using our method and compare our results with the boundary computed using a finite-difference method. As we can see, in both figures, the curves are visually indistinguishable.

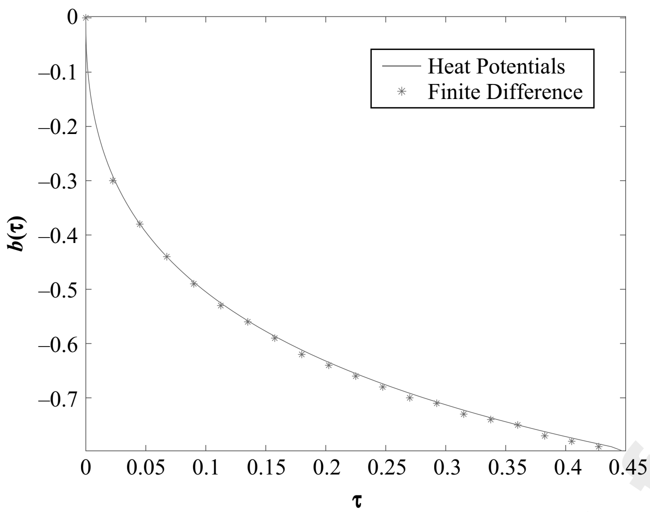
EXHIBIT 1

Parameters for an American Option

T	r	σ	K
5	0.02	0.3	100

EXHIBIT 2

Early Exercise Boundary $b(\tau)$ Computed Using the Method of Heat Potentials and Finite Difference Method in Transformed Coordinates



In Exhibit 4 we show the density function $\nu(\tau)$ computed using our method. As expected, it is monotone and positive.

Finally, we show the curve computed using our method and the exercise boundary approximation for small τ (Ostrov and Goodman 2002); see Exhibit 5. We observe that for small τ the approximation works well, while for larger τ it becomes inaccurate, as one would expect.

THE STRUCTURAL DEFAULT PROBLEM

Following Hyer et al. (1999), Hull and White (2001), and Avellaneda and Zhu (2001), we assume that the normalized asset value is driven by $X_t = W_t$ and the default occurs at time s when X_t hits the boundary $b(t)$

$$s = \inf\{t > 0 : X_t \leq b(t)\}.$$

EXHIBIT 3

Early Exercise Boundary $S_T^*(t)$ Computed in the Original Coordinates

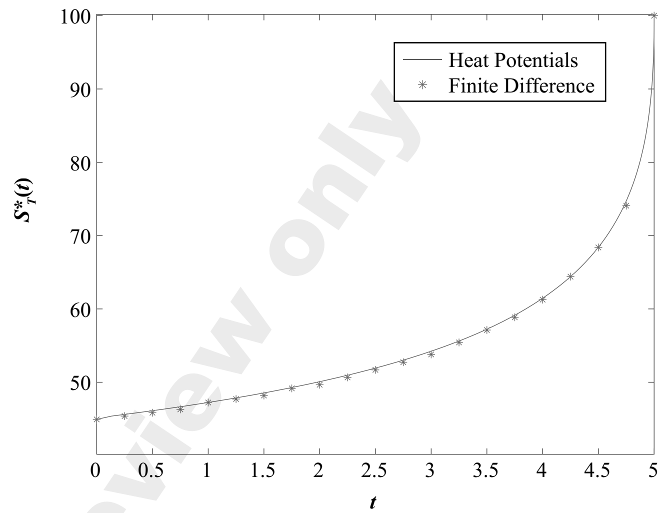
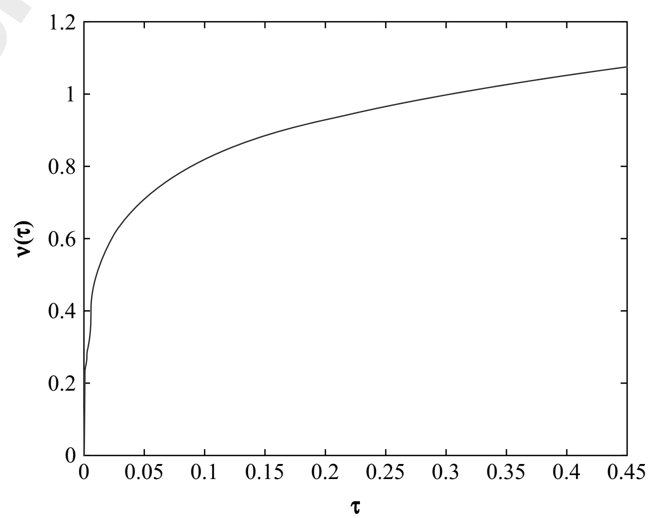


EXHIBIT 4

Density Function $\nu(\tau)$



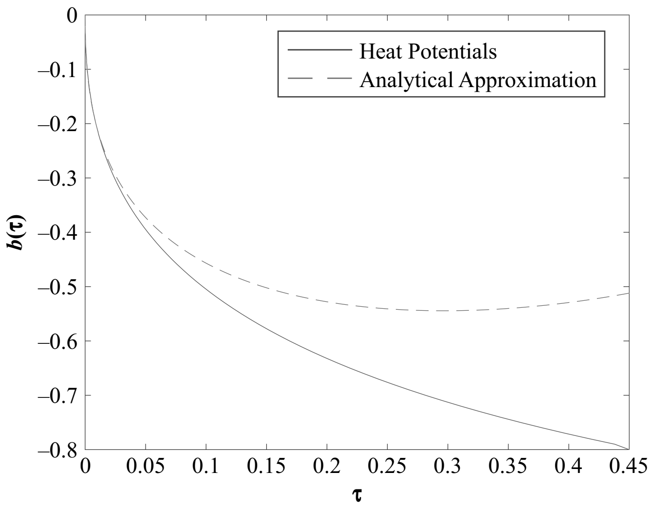
Note: The visible kink in the graph is due to our choice of τ_0.

We are going to find $b(t)$ assuming we know the default probability $\pi(t)$. For brevity, we assume that X_t has unit volatility by rescaling t as necessary.

Initially, we assume that for the initial short time period $[0, \tau]$ the firm is not going to default, and the first possibility to default occurs at time τ. Eventually, we

EXHIBIT 5

The Early Exercise Boundary Computed Using the Method of Heat Potentials and Using the Analytical Approximation of Ostrov and Goodman (2002)
$\tilde{T} = 0.45$



shall let $\tau \to 0$. From a practical standpoint, a choice of a small enough τ is sufficient, but it is rather interesting to see what happens from a theoretical standpoint. Hence, we redefine the default time as

$$s = \inf\{t \geq \tau : X_t \leq b(t)\}.$$

We denote the default probability $P(t, T, z) = \mathbb{P}(t \leq s \leq T | X_t = z)$, and $p(t, x; z)$ is the transition probability density from the state $(0, z)$ to the state (t, x). We also denote $\pi(t) = P(0, t)$.

It is clear that at time $t = \tau - 0$, the transition probability is

$$p(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}.$$

At time $t = \tau$, the first possibility of default occurs. Assuming we know the default boundary value $b_\tau = b(\tau)$, the transition probability becomes

$$p(\tau, x) = \begin{cases} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}, & x \geq b_\tau \\ 0, & x < b_\tau. \end{cases}$$

Hence, transition probability satisfies the following Fokker–Planck equation

$$p_t(t, x) = \frac{1}{2} p_{xx}(t, x), \tag{8}$$

$$p(\tau, x) = \begin{cases} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}, & x \geq b_\tau \\ 0, & x < b_\tau, \end{cases}$$

$$p(t, b(t)) = 0. \tag{9}$$

Then, the default probability density $g(t)$ can be written as

$$g(t) = \frac{1}{2} p_x(t, b(t)),$$

so that the default probability $\pi(t) = \int_0^t g(u) du$. Alternatively,

$$\pi(t) = 1 - \int_{b(t)}^{\infty} p(t, x) dx.$$

Solution Using the Method of Heat Potentials

We split (8)–(9) as $p(t, x) = H(t, x) + q(t, x)$:

$$\begin{aligned} q_t &= \frac{1}{2} q_{xx}, \\ q(\tau, x) &= 0 \\ q(t, b(t)) &= -H(t, b(t)). \end{aligned} \tag{10}$$

and

$$H_t = \frac{1}{2} H_{xx},$$

$$H(\tau, x) = \begin{cases} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}, & x \geq b_\tau \\ 0, & x < b_\tau. \end{cases}$$

Solving the last equation, as a convolution of heat kernel with the initial condition, we get

$$H(t, x) = \int_{-\infty}^{+\infty} \frac{1}{2\pi\sqrt{t-\tau}} e^{-\frac{(x-y)^2}{2(t-\tau)}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{y^2}{2\tau}} 1_{\{y \geq b_\tau\}} dy.$$

Consider

$$\int_{b_\tau}^{+\infty} e^{-\frac{(x-y)^2}{2(t-\tau)}-\frac{y^2}{2\tau}}dy = e^{\frac{\tilde{x}^2}{2u}-\frac{x^2}{2t}}\int_{b_\tau}^{+\infty} e^{-\frac{(\tilde{x}-y)^2}{2u}}dy$$
$$= \sqrt{2\pi u}e^{\frac{\tilde{x}^2}{2u}-\frac{x^2}{2t}}\left(1-N\left(\frac{b_\tau-\tilde{x}}{\sqrt{u}}\right)\right),$$

where $u=\frac{(t-\tau)\tau}{t}$, $\tilde{x}=x\frac{u}{t}$, and $N(x)$ is the standard normal CDF.

As a result,

$$H(t,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\left(1-N\left(\frac{b_\tau-\tilde{x}}{\sqrt{u}}\right)\right).$$

The IBVP (10) can be solved using the method of heat potentials (Lipton 2001, Section 12.2.3, pp. 462–467):

$$q(t,x)=\int_\tau^t \frac{(x-b(t'))\exp\left(-\frac{(x-b(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt',$$

where $\nu(t)$ solves the following Volterra integral equation of the second kind

$$\nu(t)+\int_\tau^t \frac{(b(t)-b(t'))\exp\left(-\frac{(b(t)-b(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt' + H(t,b(t))=0. \qquad (11)$$

Expressing $p(t,x)$ via $H(t,x)$ and $q(t,x)$, and computing $\pi(t)$ as

$$\pi(t)=1-\int_{b(t)}^{+\infty}p(t,x)dx,$$

we get

$$\pi(t)=1-\tilde{H}(t)-\int_\tau^t \frac{\exp\left(-\frac{(b(t)-b(t'))^2}{2(t-t')}\right)\nu(t')}{\sqrt{2\pi(t-t')}}dt', \qquad (12)$$

where

$$\tilde{H}(t)=\int_{b(t)}^{+\infty}H(t,x)dx.$$

To compute $\tilde{H}(t)$, we use the following result (formula 10,010.1 from Owen 1980):

$$\int_{-\infty}^{Y}N(a+bx)n(x)dx = BvN\left(\frac{a}{\sqrt{1+b^2}},Y;-\frac{b}{\sqrt{1+b^2}}\right).$$

Transforming $\tilde{H}(t)$, we have

$$\tilde{H}(t)=1-N\left(\frac{b(t)}{\sqrt{t}}\right)-\frac{1}{\sqrt{2\pi t}}\int_{b(t)}^{\infty}e^{-\frac{x^2}{2t}}N\left(\frac{b_\tau-\tilde{x}}{\sqrt{u}}\right)dx$$
$$=1-N\left(\frac{b(t)}{\sqrt{t}}\right)-N\left(b_\tau\sqrt{\frac{t}{u(u+t)}}\right)$$
$$+BvN\left(b_\tau\sqrt{\frac{t}{u(u+t)}},\frac{b(t)}{\sqrt{t}};\sqrt{\frac{u}{u+t}}\right),$$

so that Eq. (12) becomes

$$\pi(t)=N\left(\frac{b(t)}{\sqrt{t}}\right)+N\left(b_\tau\sqrt{\frac{t}{u(u+t)}}\right)$$
$$-BvN\left(b_\tau\sqrt{\frac{t}{u(u+t)}},\frac{b(t)}{\sqrt{t}};\sqrt{\frac{u}{u+t}}\right)$$
$$-\int_\tau^t \frac{\exp\left(-\frac{(b(t)-b(t'))^2}{2(t-t')}\right)\nu(t')}{\sqrt{2\pi(t-t')}}dt'.$$

where $BvN(x,y,\rho)$ is the CDF of bivariate normal distribution with correlation ρ.

The Choice of $b(\tau)$

Consider the default probability of the form

$$\pi(t)=1-e^{-\eta t}.$$

The barrier has to start at τ, $\tau\to 0$, and there should be no barrier before that. We wish to find $b(\tau)$. We have

$$\pi(\tau)=1-\int_{b(\tau)}^{\infty}\frac{\exp\left(-\frac{x^2}{2\tau}\right)}{\sqrt{2\pi\tau}}dx=1-N\left(-\frac{b(\tau)}{\sqrt{\tau}}\right)=1-e^{-\eta\tau}.$$

Thus,

$$N\left(-\frac{b(\tau)}{\sqrt{\tau}}\right)=e^{-\eta\tau},$$

and

$$b(\tau) = -\sqrt{\tau} N^{-1}(e^{-\eta\tau}).$$

Now, using Blair et al. (1976)

$$N^{-1}(\gamma) \underset{\gamma \to 1}{\sim} \sqrt{2 f(\zeta)},$$

where

$$\zeta = -\ln(2\sqrt{\pi}(1-\gamma)),$$
$$f(\zeta) = \zeta - \frac{\ln \zeta}{2} + \frac{\ln \zeta - 2}{4\zeta} + \frac{(\ln \zeta)^2 - 6 \ln \zeta + 14}{16\zeta^2},$$

so that

$$\begin{aligned} b(\hat{\tau}) &= -\sqrt{2\hat{\tau} f(-\ln(2\sqrt{\pi}(1-e^{-\eta\hat{\tau}})))} \\ &\approx -\sqrt{2\hat{\tau}\left|\ln(2\sqrt{\pi}\eta\hat{\tau})\right|}. \end{aligned} \quad (13)$$

Once again, we can think of $b(\tau)$ as a typical square-root-like boundary.

Numerical Method

We assume that the default boundary $b(t)$ is piecewise linear:

$$b(t_i) = b(\tau) + \sum_{j=1}^{i} \gamma_j (t_j - t_{j-1}), \quad (14)$$

and on each step i we determine the corresponding γ_i to match the default probability $\pi(t_i)$.

The numerical method is the following:

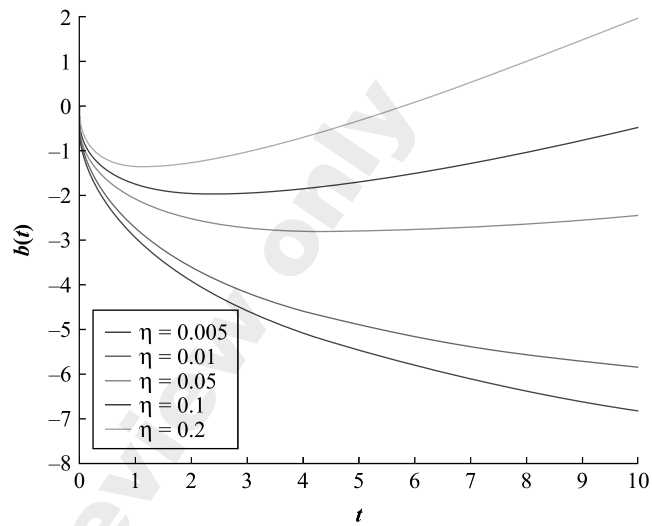
Algorithm 2 Numerical Method

Determine b_τ using (13).
for $i = 1 : n$ **do**
Compute (14) as a function of γ_i.
Find $v(t)$ as a solution of (11).
Solve (12) with respect to unknown γ_i.
Compute $b(t_i)$.
end for

The Volterra integral equation (11) could be solved by using a quadrature approximation, an example of such

EXHIBIT 6

Default Boundary $b(t)$ Computed Using Numerical Method for Different η



a method can be found in Lipton and Kaushansky (2019), after that, the integrals in (12) can be easily computed.

Numerical Results

Consider $T = 10$ and a non-uniform grid $t_i = (i\Delta)^2$ for $i = 0, \ldots N$. This choice is motivated by having more points on the short end and sparser grid on the long end. We choose $N = 500$. To solve the Volterra equation (11), we consider different grids with a smaller step size.

We apply the methods described in the previous section and analyze the results. As before, we assume that market default probability is given by a parametric family

$$\pi(t) = 1 - e^{-\eta t}, \quad \eta > 0.$$

In Exhibit 6, we show the default boundaries computed by using our method for different values of η.

We also plot the solution of the Volterra equation (11) in Exhibit 7.

To verify correctness of out method, we take the default boundary computed before and compute the default probability using the Monte Carlo method. In Exhibit 8, we compare the results and see that the curves are visually indistinguishable.

EXHIBIT 7

Density Functions $\nu(t)$

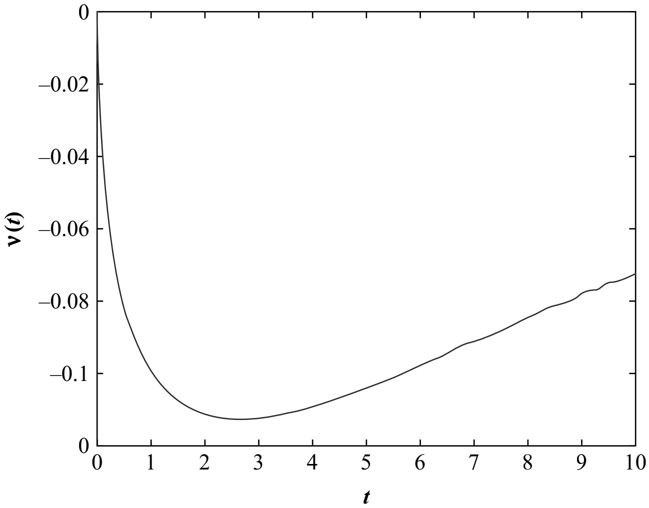
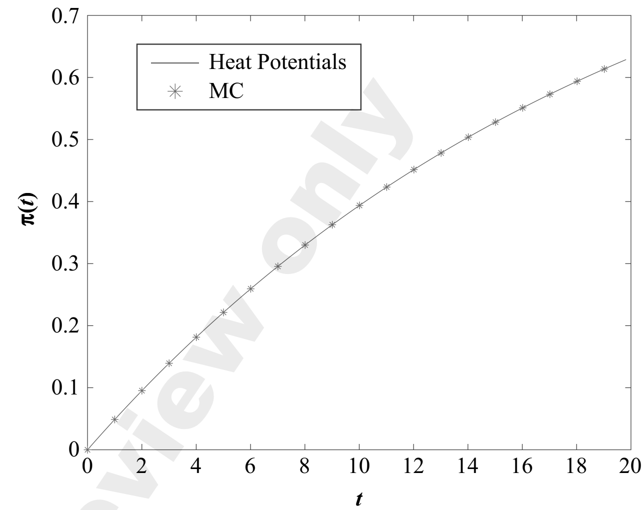


EXHIBIT 8

Default Probabilities Computed Using the MC Method and the Method of Heat Potentials



FIRST HITTING TIME DENSITY FOR AN ORNSTEIN–UHLENBECK PROCESS

Consider an Ornstein–Uhlenbeck process with time-dependent coefficients

$$\begin{aligned} dX_t &= \lambda(t)(\theta(t) - X_t)dt + \sigma(t)dW_t, \\ X_0 &= z. \end{aligned} \tag{15}$$

We wish to calculate the density of stopping time $s = \inf\{t : X_t = b(t)\}$ for some time-dependent barrier $b(t)$.

Transformation to a standard OU process Lipton and Kaushansky (2019) showed how to transform the process to a standard OU process

$$d\bar{X}_{\bar{t}} = -\bar{X}_{\bar{t}}\, d\bar{t} + d\bar{W}_{\bar{t}}.$$

We state their result in the following lemma

Lemma 1 *Consider*

$$\bar{X}_{\bar{t}} = p(\bar{t})X_t + q(\bar{t}),$$

where

$$\begin{aligned} p(t) &= e^{\Lambda(t)-M(t)}, \\ q(t) &= e^{-M(t)}\left(-\int_0^t \lambda(u)\theta(u)e^{\Lambda(u)}du\right), \\ \bar{t}(t) &= M(t), \\ \Lambda(t) &= \int_0^t \lambda(u)du, \end{aligned}$$

$$M(t) = \frac{1}{2}\ln\left(2\int_0^t \sigma^2(u)e^{2\Lambda(u)}du + 1\right),$$

and X_t satisfies (15). Then, $\bar{X}_{\bar{t}}$ is a standard OU process.

As a simple corollary, for the process with constant coefficients $\lambda(t) \equiv \lambda$, $\theta(t) \equiv \theta$, and $\sigma(t) \equiv \sigma$, we have

$$\begin{aligned} \bar{X}_{\bar{t}} &= \frac{\sqrt{\lambda}}{\sigma}(X_t - \theta), \\ \bar{t} &= \lambda t. \end{aligned}$$

Hence, in the following, we consider the hitting problem for a standard Ornstein–Uhlenbeck process.

Transformation of a standard OU process to a Wiener process (Forward) Consider $z > b(0)$. To calculate the density of the hitting time distribution $g(t, z)$, we need to solve the following forward problem

$$\begin{aligned} p_t(t,x;z) &= p(t,x;z) + xp_x(t,x;z) \\ &\quad + \frac{1}{2}p_{xx}(t,x;z), \\ p(0,x;z) &= \delta(x-z), \\ p(t,b(t);z) &= 0. \end{aligned}$$

This distribution is given by

$$g(t,z) = \frac{1}{2}p_x(t,b(t);z).$$

Introducing new variables

$$q(\tau,\xi) = e^{-t}p(t,x), \quad \tau = e^{t}\sinh(t), \quad \xi = e^{t}x,$$

We get

$$\begin{aligned} q_{\tau}(\tau,\xi) &= \frac{1}{2}q_{\xi\xi}(\tau,\xi), \\ q(0,\xi) &= \delta(\xi - z), \\ q(\tau,\beta(\tau)) &= 0, \\ \beta(\tau) &= \sqrt{2\tau+1}\tilde{b}(t), \end{aligned} \tag{16}$$

where $\tilde{b}(\tau) = b(t(\tau))$.

It is clear that $0 \le \tau < \infty$ and $e^{t} = \sqrt{2\tau+1}$.

Then,

$$g(\tau) = \frac{1}{2}q_{\xi}(\tau,\beta(\tau)), \tag{17}$$

The hitting density (17) can be found using the method of heat potentials by writing the solution for $q(\tau, \xi)$, differentiating it over ξ, and computing the limit at $\beta(\tau)$. Using Theorem 1 from Lipton and Kaushansky (2019), we can write the final formula for the hitting density

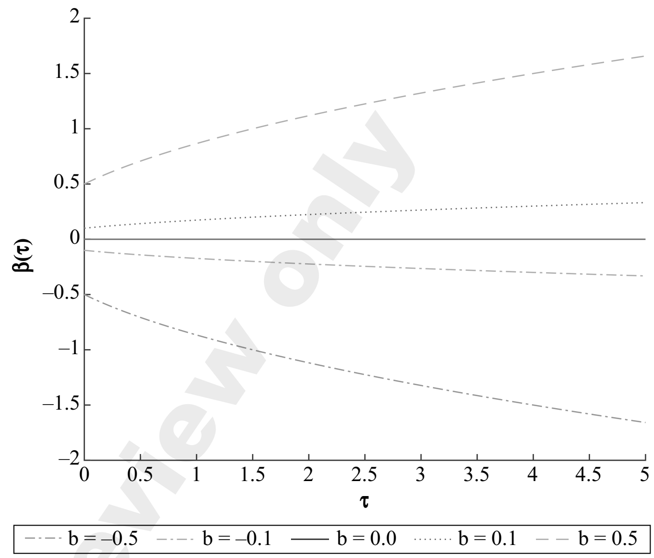
$$\begin{aligned} g(t) = &-\frac{(e^{t}b(t)-z)\exp\left(-\frac{(e^{t}b(t)-z)^2}{(e^{2t}-1)}+2t\right)}{\sqrt{\pi(e^{2t}-1)^3}} \\ &-\left(e^{t}b(t)+\frac{e^{2t}}{\sqrt{\pi(e^{2t}-1)}}\right)\nu(t)+\frac{e^{2t}}{\sqrt{8\pi}} \\ &\int_0^{\tau}\frac{\left(1-\frac{(\beta(\tau)-\beta(\tau))'^2}{(\tau-\tau')}\right)\exp\left(-\frac{(\beta(\tau)-\beta(\tau'))^2}{2(\tau-\tau')}\right)\nu(\tau')-\nu(\tau)}{\sqrt{(\tau-\tau')^3}}d\tau', \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\nu(\tau)+\int_0^{\tau}\frac{(\beta(\tau)-\beta(\tau'))\exp\left(-\frac{(\beta(\tau)-\beta(\tau'))^2}{2(\tau-\tau')}\right)\nu(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau' \\ &+\frac{\exp\left(\left(-\frac{(\beta(\tau)-z)^2}{2\tau}\right)\right)}{\sqrt{2\pi\tau}} = 0. \end{aligned} \tag{19}$$

Exhibit 9

Moving Boundary β(τ) for (16)



An example of the transformed boundary, for the case of the flat boundary, is given in Exhibit 9. **Transformation of a Standard OU Process to a Wiener Process (Backward).**

Alternatively, we can solve the backward problem:

$$\begin{aligned} G_t(t,T,z) &= -zG_z(t,T,z)+\frac{1}{2}G_{zz}(t,T,z), \\ G(0,T,z) &= 0, \\ G(t,T,b(T-t)) &= 1, \end{aligned} \tag{20}$$

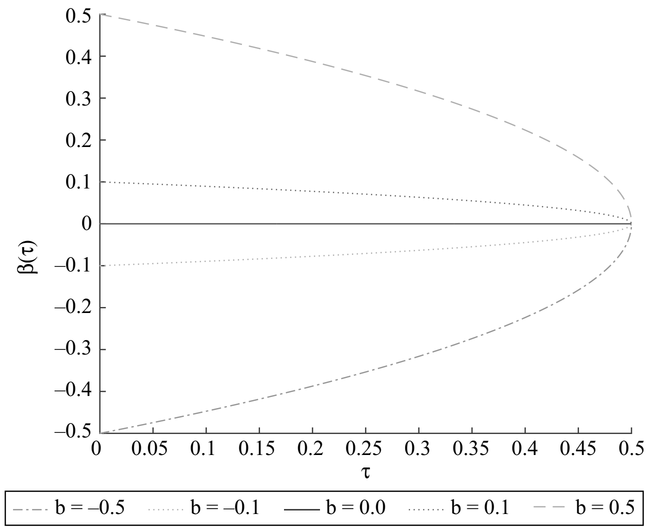
By the same token as before, we introduce

$$\begin{aligned} &\tau = \varpi(t) = \frac{1-e^{-2t}}{2} = e^{-t}\sinh(t), \\ &\hat{T} = e^{-T}\sinh(T), \quad 0 \le \tau < \frac{1}{2}, \quad x = e^{-t}z, \end{aligned}$$

and rewrite (20) as follows

$$\begin{aligned} G_{\tau}(\tau,x) &= \frac{1}{2}G_{xx}(\tau,x), \\ G(0,x) &= 0, \\ G(\tau,\beta(\hat{T}-\tau)) &= 1, \\ \beta(\tau) &= \sqrt{1-2\tau}b(T-t), \end{aligned} \tag{21}$$

EXHIBIT 10
Moving Boundary $\beta(\tau)$ for (21)



It is clear that $0 \le \tau < 1/2$, and $e^{-t} = \sqrt{1-2\tau}$. It is worth noting that for the backward problem the computational domain is compactified in the τ direction.

The backward problem is particularly useful when $b(t) \equiv b$ since the problem does not depend on T in this case. We show the moving boundary for different values of b in Exhibit 10.

Using the method of heat potentials, Lipton and Kaushansky (2019) (Theorem 2) derived the expression for $G(\tau, x)$:

$$G(\tau,x) = \int_0^{\tau} \frac{(x - b\sqrt{1-2\tau'})\exp\left(-\frac{(x-b\sqrt{1-2\tau'})^2}{2(\tau-\tau')}\right)\nu^b(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}\, d\tau', \tag{22}$$

where

$$\nu^b(\tau) - \sqrt{\frac{2}{\pi}}\, b \int_0^{\tau} \frac{\exp\left(-b^2 \frac{(\sqrt{1-2\tau'}-\sqrt{1-2\tau})}{(\sqrt{1-2\tau'}+\sqrt{1-2\tau})}\right)\nu^b(\tau')}{(\sqrt{1-2\tau'}+\sqrt{1-2\tau})\sqrt{\tau-\tau'}}\, d\tau' - 1 = 0. \tag{23}$$

Once (23) is solved, $G(\tau, x)$ and $G(t, z)$ can be calculated by virtue of (22) in a straightforward fashion.

Numerical Results

Equations (18), (19), (22), and (23) can be solved using numerical methods from Lipton and Kaushansky (2018) and Lipton and Kaushansky (2019).

In this section, we consider four examples. First, we consider the case of the standard OU process with a flat boundary. Next, we consider two examples of time-dependent periodic boundaries, and finally we consider the boundary of the form $b(t) = Ae^{-t} + Be^{t}$, for which the closed-form solution is available.

In our first example, we choose $T = 2$, $z = 5$, and consider a flat barrier $b = 2$ for the standard OU process.

In Exhibit 11, we show the hitting density and corresponding CDF. In Exhibit 12, we show the density functions $\nu(t)$.

Next, we consider an example with a non-flat periodic boundary. Consider $T = 2$, $b(t) = 0.01\sin 10t$, and $z = 2$. The boundary is a periodic function, which fluctuates around 0. We compare the results with $b \equiv 0$.

In Exhibit 13, we show the hitting density and corresponding CDF. We can see that there is a small difference in $g(t)$, while $G(t)$ are visually indistinguishable.

Next, we consider the same parameters with $b(t) = -0.1\sin 10t$. In this case, the fluctuations are more significant and have a larger impact on $g(t)$ and $G(t)$. The results are given in Exhibit 14.

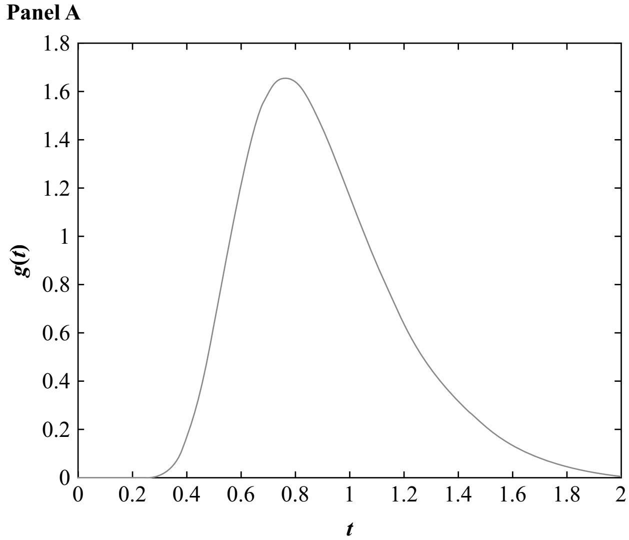
Next, we consider $T = 1$, $z = 1$, and $b(t) = 0.1e^{-t} - 0.1e^{t}$. As discussed in Lipton and Kaushansky (2019), after the change of variables, this problem transforms to the problem for Brownian motion with a linear boundary, and can be easily solved analytically. In Exhibit 15, we compare our results with the closed-form solution and observe that they are visually indistinguishable.

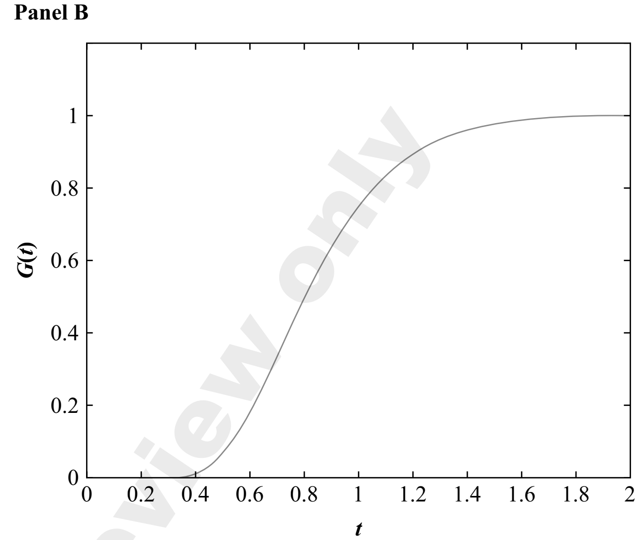
CONCLUSION

In this article, we have demonstrated the power of the method of heat potentials by solving three classical problems of financial mathematics: (A) pricing of the American put, (B) calibrating the default boundary in a structural default model, and (C) describing the first hitting time for an Ornstein–Uhlenbeck process. For all three problems, we found a semi-analytical solution. In all cases we reduced the original problem to the IBVP for a heat equation with a moving boundary, and applied the method of heat potentials to obtain a coupled system of Volterra integral equations of the second kind. The

E X H I B I T 11

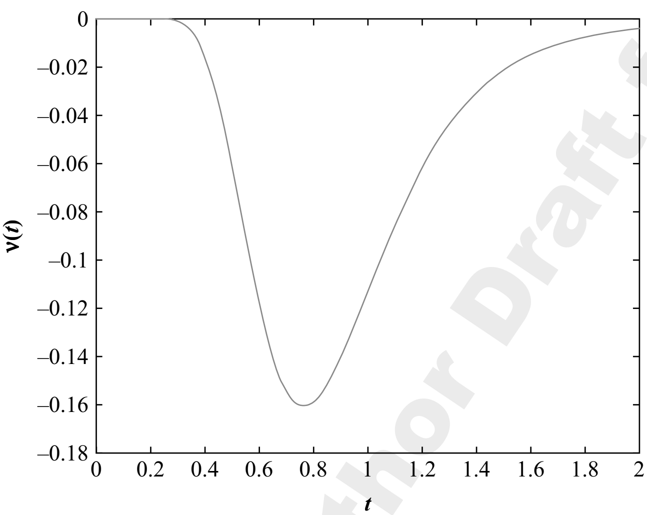
(A) Density Function $g(t)$; (B) Cumulative Density Function $G(t)$ for a Flat Boundary





E X H I B I T 12

Density Function $\nu(t)$ for a Flat Boundary



resulting system is much easier to solve numerically than the original problem.

For all three problems, we developed numerical methods for solving the coupled system of Volterra integral equations, and compared our solutions with solutions obtained by other known methods.

Our results clearly show that there are a lot of problems in mathematical finance, which can be efficiently solved using the method of heat potentials. Other problems will be examined elsewhere.

One novel result worth mentioning is that a curvilinear square-root-like default boundary capable of producing nontrivial default probability for the structural default model does exist.

EXHIBIT 13

(A) Density Function; (B) Cumulative Density Function $g(t)$ for $b(t) = 0.01\sin 10t$

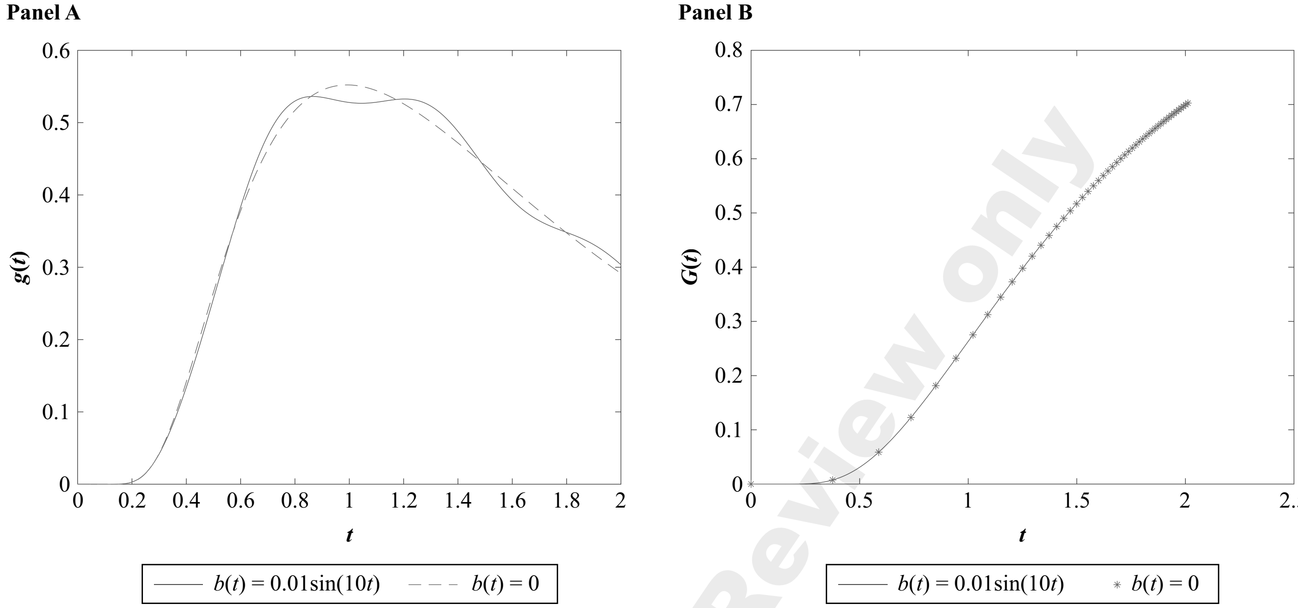


EXHIBIT 14

(A) Density Function $g(t)$; (B) Cumulative Density Function $g(t)$ for $b(T) = -0.1\sin 10t$

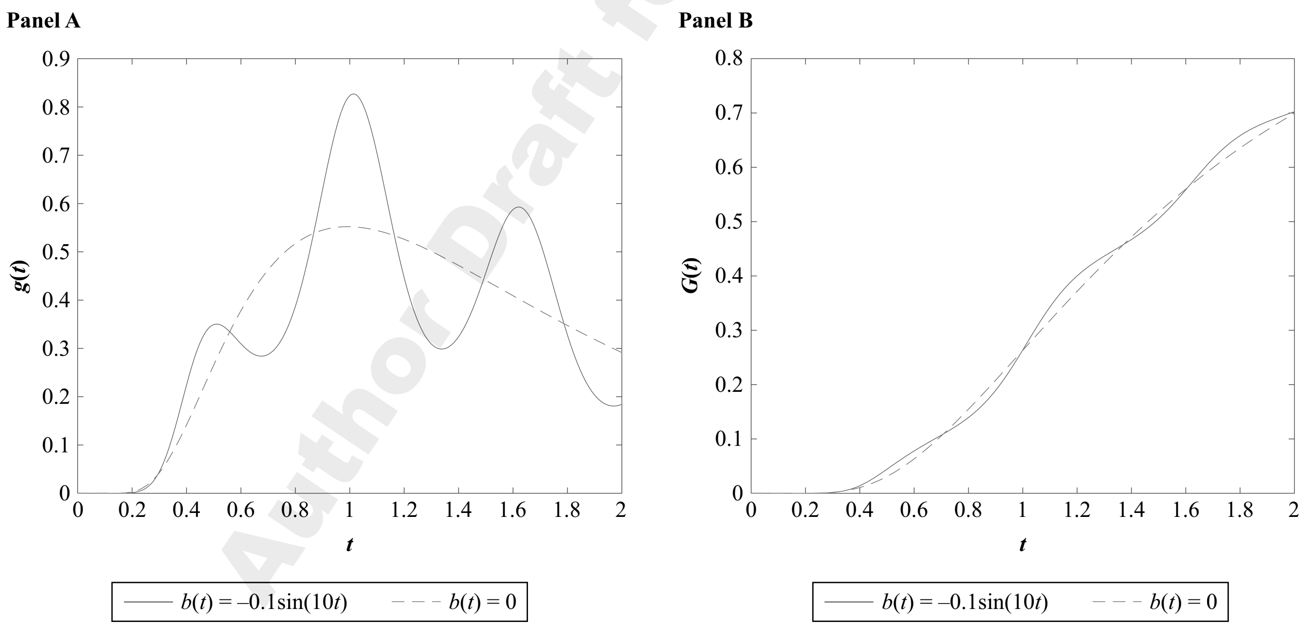
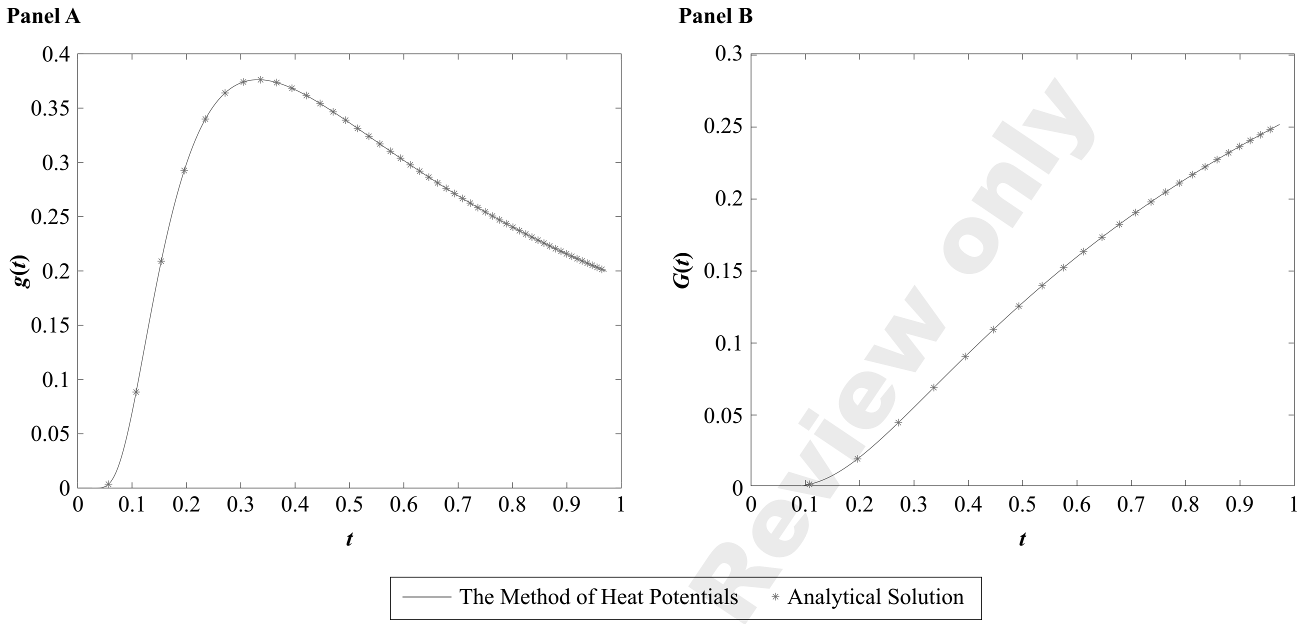


EXHIBIT 15

The Case with Time-Dependent Boundary $b(t) = 0.1e^{-t} - 0.1e^{t}$: Analytical and Numerical Solutions (A) Density Function $g(t)$; (B) CDF $G(t)$



APPENDIX A

DERIVATION OF THE LIMITS IN THE METHOD OF HEAT POTENTIALS

Let $u(t,x)$ be defined as follows:

$$u(t,x) = \int_0^t \frac{(x-b(t'))\exp\left(-\frac{(x-b(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt'.$$

We need to calculate $u(t, b(t))$, or the limit of $u(t, b(t)+\varepsilon)$ when $\varepsilon \to +0$. We also need to calculate $u_x(t, b(t))$, or the limit of $u_x(t, b(t)+\varepsilon)$ when $\varepsilon \to +0$.

To this end, we write

$$\begin{aligned}u(t,b(t)) &= \lim_{\varepsilon\to 0} u(t,b(t)+\varepsilon)\\ &= \lim_{\varepsilon\to 0}\int_0^t \frac{(\varepsilon+\Psi(t,t'))\exp\left(-\frac{(\varepsilon+\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt'\\ &= \mathbb{M}^{(1)} + \mathbb{M}^{(2)},\end{aligned}$$

where

$$\mathbb{M}^{(1)} = \lim_{\varepsilon\to 0}\varepsilon\int_0^t \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} - \varepsilon\frac{\Psi(t,t')}{(t-t')} - \frac{\Psi(t,t')^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt',$$

$$\mathbb{M}^{(2)} = \lim_{\varepsilon\to 0}\int_0^t \frac{\Psi(t,t')\exp\left(-\frac{\varepsilon^2}{2(t-t')} - \varepsilon\frac{\Psi(t,t')}{(t-t')} - \frac{\Psi(t,t')^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt'.$$

Since the second integrand has integrable singularity, we have

$$\mathbb{M}^{(2)} = \int_0^t \frac{\Psi(t,t')\exp\left(-\frac{\Psi(t,t')^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt'.$$

We can drop non-singular terms approaching unity and simplify the first limit as follows:

$$\mathbb{M}^{(1)} = \lim_{\varepsilon\to 0}\varepsilon\int_0^t \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} - \frac{\Psi(t,t')^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}}\nu(t')dt'.$$

We can write

$$\nu(t') = \nu(t) + \left(\exp\left(-\frac{\Psi(t,t')^2}{2(t-t')} \right) \nu(t') - \nu(t) \right),$$

and split the above integrand into a singular part and a part with integrable singularity, which vanishes in the limit, so that

$$\mathbb{M}^{(1)} = \nu(t) \lim_{\varepsilon\to 0} \varepsilon \int_0^t \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} \right)}{\sqrt{2\pi(t-t')^3}} dt'.$$

The first change of variables $(t-t')/\varepsilon^2 \to u$ yields:

$$\mathbb{M}^{(1)} = \nu(t) \lim_{\varepsilon\to 0} \int_0^{t/\varepsilon^2} \frac{\exp\left(-\frac{1}{2u} \right)}{\sqrt{2\pi u^3}} du.$$

The second change of variables $u \to 1/\nu^2$ yields:

$$\mathbb{M}^{(1)} = 2\nu(t) \lim_{\varepsilon\to 0} \int_{\varepsilon/\sqrt{t}}^{\infty} \frac{\exp\left(-\frac{v^2}{2} \right)}{\sqrt{2\pi}} d\nu = \nu(t).$$

Finally,

$$u(t,b(t)) = \nu(t) + \int_0^t \frac{(b(t)-b(t'))\exp\left(-\frac{(b(t)-b(t'))^2}{2(t-t')} \right)}{\sqrt{2\pi(t-t')^3}} \nu(t')dt'.$$

We analyze the limit $\lim_{\varepsilon\to 0} u_x(t, b(t) + \varepsilon)$ by the same token. We have

$$\begin{aligned} u_x(t,b(t)) &= \lim_{\varepsilon\to 0} u_x(t,b(t)+\varepsilon) \\ &= \lim_{\varepsilon\to 0} \left(\int_0^t \frac{\left(1-\frac{(\varepsilon+\Psi(t,t'))^2}{(t-t')}\right)\exp\left(-\frac{(\varepsilon+\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi(t-t')^3}} \nu(t')dt' \right) \\ &= \lim_{\varepsilon\to 0} \left(\begin{array}{c} \int_0^t \left(1 - \frac{\varepsilon^2}{(t-t')} - 2\varepsilon \frac{\Psi(t,t')}{(t-t')} - \frac{\Psi(t,t')^2}{(t-t')} \right) \\ \times \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} - \varepsilon \frac{\Psi(t,t')}{(t-t')} \right) \Xi(t,t')}{\sqrt{2\pi(t-t')^3}} \nu(t')dt' \end{array} \right). \end{aligned}$$

We can drop non-singular terms approaching unity and simplify the above formula as follows:

$$\begin{aligned} u_x(t,b(t)) &= \lim_{\varepsilon\to 0} \int_0^t \left(1 - \frac{\varepsilon^2}{(t-t')} - 2\varepsilon \frac{\Psi(t,t')}{(t-t')} - \frac{\Psi(t,t')^2}{(t-t')} \right) \\ &\quad \times \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} \right) \Xi(t,t')}{\sqrt{2\pi(t-t')^3}} \nu(t')dt' \\ &= \mathbb{L}^{(1)} + \mathbb{L}^{(2)} - 2\mathbb{L}^{(3)} - \mathbb{L}^{(4)}, \end{aligned}$$

where

$$\mathbb{L}^{(1)} = \nu(t) \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{t/\varepsilon^2} \left(1 - \frac{1}{u} \right) \frac{\exp\left(-\frac{1}{2u} \right)}{\sqrt{2\pi u^3}} du.$$

$$\mathbb{L}^{(2)} = \lim_{\varepsilon\to 0} \int_0^t \left(1 - \frac{\varepsilon^2}{(t-t')} \right) \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} \right) (\Xi(t,t')\nu(t') - \nu(t))}{\sqrt{2\pi(t-t')^3}} dt',$$

$$\mathbb{L}^{(3)} = \lim_{\varepsilon\to 0} \varepsilon \int_0^t \frac{\Psi(t,t')}{(t-t')} \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} \right) \Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')^3}} dt',$$

$$\mathbb{L}^{(4)} = \lim_{\varepsilon\to 0} \int_0^t \frac{\Psi(t,t')^2}{(t-t')} \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} \right) \Xi(t,t')\nu(t')}{\%\sqrt{2\pi(t-t')^3}} dt'.$$

Consider $\mathbb{L}^{(1)}$. The first change of variables yields:

$$\mathbb{L}^{(1)} = \nu(t) \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{t/\varepsilon^2} \left(1 - \frac{1}{u} \right) \frac{\exp\left(-\frac{1}{2u} \right)}{\sqrt{2\pi u^3}} du.$$

The second change of variables yields

$$\begin{aligned} \mathbb{L}^{(1)} &= \frac{2\nu(t)}{\sqrt{2\pi}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\varepsilon/\sqrt{t}}^{\infty} (1-\nu^2) \exp\left(-\frac{\nu^2}{2} \right) dv \\ &= -\frac{2\nu(t)}{\sqrt{2\pi}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon/\sqrt{t}} (1-\nu^2) \exp\left(-\frac{\nu^2}{2} \right) dv \\ &= -\frac{2\nu(t)}{\sqrt{2\pi t}}. \end{aligned}$$

Here we use the fact that

$$\int_0^{\infty} (1-\nu^2) \exp\left(-\frac{\nu^2}{2} \right) dv = 0.$$

Next,

$$\mathbb{L}^{(2)} = \mathbb{L}_1^{(2)} - \mathbb{L}_1^{(2)},$$

$$\mathbb{L}_1^{(2)} = \lim_{\varepsilon\to 0}\int_0^t \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')}\right)(\Xi(t,t')\nu(t') - \nu(t))}{\sqrt{2\pi(t-t')^3}}dt'$$
$$= \int_0^t \frac{(\Xi(t,t')\nu(t') - \nu(t))}{\sqrt{2\pi(t-t')^3}}dt',$$

because the singularity is integrable,

$$\mathbb{L}_2^{(2)} = \lim_{\varepsilon\to 0}\varepsilon^2\int_0^t \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')}\right)(\Xi(t,t')\nu(t') - \nu(t))}{(t-t')\sqrt{2\pi(t-t')^3}}dt' = 0,$$

in view of our derivation of the expression for $\mathbb{L}^{(1)}$. The next integral can be computed by the same token as $\mathbb{L}^{(1)}$,

$$\mathbb{L}^{(3)} = \lim_{\varepsilon\to 0}\varepsilon\int_0^t \frac{\Psi(t,t')}{(t-t')}\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')}\right)\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')^3}}dt'$$
$$= \lim_{\varepsilon\to 0}\varepsilon\int_0^t \frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')}\right)\Omega(t,t')}{\sqrt{2\pi(t-t')^3}}dt',$$

where

$$\Omega(t,t') = \frac{\Psi(t,t')\Xi(t,t')\nu(t')}{(t-t')}.$$

Accordingly, we get

$$\mathbb{L}^{(3)} = \Omega(t,t) = b'(t)\nu(t).$$

Finally, the integrand in $\mathbb{L}^{(4)}$ has integrable singularity, so

$$\mathbb{L}^{(4)} = \int_0^t \frac{\Psi(t,t')^2}{(t-t')}\frac{\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')^3}}dt'.$$

Combining all the terms, we get

$$u_x(t,b(t)) = -2\left(\frac{1}{\sqrt{2\pi t}} + b'(t)\right)\nu(t)$$
$$+ \int_0^t \frac{\left(\left(1 - \frac{\Psi(t,t')^2}{(t-t')}\right)\Xi(t,t')\nu(t') - \nu(t)\right)}{\sqrt{2\pi(t-t')^3}}dt'.$$

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ADDITIONAL READING

Valuing Credit Default Swaps I

No Counterparty Default Risk
John C. Hull and Alan D White
The Journal of Derivatives
https://jod.pm-research.com/content/8/1/29

ABSTRACT: *One of the fastest growing areas of both derivatives trading and research right now is in contracts based on credit risk. The credit default swap is a standard instrument, offering the possibility of hedging against default by the issuer of an underlying bond. Several existing valuation methodologies differ in their assumptions about the payoff in case of a credit event. In this article, Hull and White present an approach based on the realistic assumption that the amount bondholders will claim in a default is based on the difference between the bond&'s post-default market value and its face value. An important contribution of this article is to use the term structure of risk-neutral implied default probabilities obtained from market prices for a set of bonds of the same issuer. The dependence of swap values on assumed recovery rates and the shape of the yield curve are explored.*

Valuing Credit Default Swaps II

Modeling Default Correlations
John C Hull and Alan D White
The Journal of Derivatives
https://jod.pm-research.com/content/8/3/12

ABSTRACT: *"In the Fall 2000, Journal of Derivatives, Hull and White presented a model for pricing credit default swaps based on the realistic assumption that in a default the bondholders will claim the difference between the bond&'s post-default market value and its face value. An important feature of the approach is the use of market prices for a set of bonds from the same issuer to obtain a term structure of risk-neutral implied default probabilities. This article extends the model significantly to allow for the existence of multiple correlated default risks. Correlations are important either when the swap is subject to counterparty credit risk, or when there are multiple underlyings with correlated risks, as in a basket default swap."*